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MULTIPLE STAGE OPTIMAL CONTROL PROBLEMS WITH APPLICATIONS TO PLASMA HEATING BY NEUTRAL INJECTION

KEN TOMIYAMA

UCLA • SCHOOL OF ENGINEERING AND APPLIED SCIENCE

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Three classes of multiple stage optimal control problems are studied. The derived results are applied to the problem of toroidal plasma heating by means of neutral injection.

First a two-stage optimal control problem having an integral cost functional, whose integrand changes its form at an unspecified switching time, is considered. An existence theorem and two sets of necessary → next page

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conditions for an optimal control and switching time are derived for a general two-stage problem.

The two-stage problem is then generalized to a multi-stage problem where the integrand of the cost functional can change from one form to another among N given forms at any instant for any number of times. This problem is shown to be reducible to a standard optimal control problem by introducing a set of auxiliary control variables. It is shown that a chattering control may be encountered due to the nonconvexity of the augmented control constraint set.

Next, the fixed k-stage optimal control problem is examined. For this problem, the control is kept constant on each of the k prescribed subintervals on which the cost functional assumes a different form. This leads to an equivalent parameter optimization problem in k-dimensional Euclidean space. A feature of this finite dimensional formulation is that no approximation errors are introduced in the discretization of the system equations.

The problem of minimum input energy plasma heating by neutral injection is studied utilizing the derived results. This problem is formulated as a two-stage optimal control problem. It is shown that optimal heating is achieved by an on-off neutral injection program which is characterized by a three point boundary value problem. The results on the fixed k-stage problem are also utilized in characterizing the optimal piecewise constant neutral injection program.

The optimal heating problem is reformulated as a multi-stage optimal control problem using a single-temperature model of the plasma. The optimal heating program is shown to assume one of three possible on-off forms, depending on the heating time duration.

Finally, the stability of two classes of ion temperature feedback control systems is discussed. The results suggest the possibility of regulation of the ion temperature using feedback-controlled neutral injection.

MULTIPLE STAGE OPTIMAL CONTROL PROBLEMS WITH APPLICATIONS TO PLASMA HEATING BY NEUTRAL INJECTION

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Three classes of multiple stage optimal control problems are studied. The derived results are applied to the problem of toroidal plasma heating by means of neutral injection.

First a two-stage optimal control problem having an integral cost functional, whose integrand changes its form at an unspecified switching time, is considered. An existence theorem and two sets of necessary conditions for an optimal control and switching time are derived for a general two-stage problem.

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The system of referencing used in this dissertation is as follows: "from item 2.3" refers to an item numbered 2.3 within the same chapter, and "from item V.2.3" refers to an item numbered 2.3 in Chapter V. "Substituting (3.4)" refers to an equation (3.4) within the same chapter, and "substituting (II.3.4)" refers to an equation (3.4) in Chapter II.

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CHAPTER I

INTRODUCTION

This dissertation consists of two parts. Part 1 is devoted to the mathematical aspects of a class of optimal control problems which are motivated by the problem of optimal heating of toroidal plasmas by means of neutral beam injection. In Part 2, the derived mathematical results are applied to plasma heating problems.

1.1. Optimal Heating Problem of Plasma by Means of Neutral Injection

In toroidal plasma devices such as Tokamak, the plasma is confined by a strong toroidal magnetic field supplemented by a poloidal field produced by the plasma current. At the same time this plasma current raises the plasma temperature through the Joule heating effect. There has been some hope that Joule heating alone would be sufficient to raise the ion temperature to a high level such that a relatively low power complementary heating source would be enough to achieve the ignition temperature. However, from the recent experiments, it has become apparent that Joule heating is insufficient for this task and auxiliary heating such as neutral injection may have to play a dominant role in plasma heating. Since the energy consumption due to auxiliary heating may become comparable to that due to Joule heating, it is important to operate the heating system at a high level of efficiency.

In this dissertation, several optimal control problems motivated by the heating problems of toroidal plasma devices by means of neutral injection are studied.

The principle of neutral beam injection heating is the following: First, the energetic neutral beam is produced by neutralizing the accelerated ion beam so that the beam can penetrate the confining magnetic field. The injected neutral particles then charge-exchange with the plasma ions to produce fast (hot) plasma ions and slow (cold) neutral particles. In a plasma heating experiment, neutral injection is introduced when the plasma temperatures and densities are built up to a sufficiently high level. It is important to know when neutral injection should be initiated in order to minimize the total input energy while achieving the desired ion temperature within a given time duration. Suppose the experiment starts at time $t = 0$ and lasts until time t_f, and neutral injection is introduced at time $t_1 \in [0,t_f]$. Then the total energy consumption can be expressed as

$$J(\text{total energy}) = \int_0^{t_f} (\text{Joule heating})dt + \int_{t_1}^{t_f} (\text{neutral injection})dt. \tag{1.1}$$

The problem of minimizing (1.1) with respect to t_1 as well as the neutral injection program leads to an optimal control problem with a two-stage cost functional J given by

$$J = \int_{t_0}^{t_1} L_1(x,u)dt + \int_{t_1}^{t_f} L_2(x,u)dt. \tag{1.2}$$

This problem will be referred to as a two-stage optimal control problem and is one of the main problems considered in this dissertation.

Another optimal heating problem arises from an engineering requirement that the neutral beam injection program be such that the beam current is kept constant on each prescribed subinterval of the

experiment time interval $[0,t_f]$. This is due to the fact that it is difficult to vary the beam current continuously in time. Moreover, the changes of the current at arbitrary time instants are not easily implemented. This motivates a fixed k-stage optimal control problem with piecewise constant controls and a cost functional which takes on k different forms on k given subintervals.

1.2. Multiple Stage Optimal Control Problems

The distinctive characteristic of the two-stage optimal control problem is that the cost functional assumes different forms before and after the unspecified switching time t_1. To solve this problem, we need to specify not only the control u but also the switching time t_1. Two necessary conditions for an optimal pair (u^*,t_1^*) are derived, one by means of calculus of variations and the other by decomposing this problem to two standard optimal control problems.

The two-stage problem is generalized to the multi-stage optimal control problem such that the integrand of the cost functional can change from one to another among N given forms any number of times. This problem is solved by reducing it to a standard problem with the help of auxiliary control variables.

There are some works on optimal control problems having system equations with discontinuous right hand side and similar problems [10], [11], [12], [13], [24], [25], [33], [41]. However, none of the above works includes the treatment of a variable intermediate switching as a part of the control. Kleinman, Fortman and Athans [26] mentioned the problem of choosing the switching times as a subject of future research in their paper on a piecewise constant feedback

control. In a paper by Athans on the optimal measurement strategies [1], he adopted a technique similar to the auxiliary controls discussed here. However, he did not discuss the possibility of chattering controls which is important in considering the optimal multi-stage controls.

The fixed k-stage optimal control problem is solved by reformulating it as a parameter optimization problem in a real k-dimensional Euclidean space E^k. Then the techniques of mathematical programming are applied to obtain a characterization of the optimal solutions.

The optimal control problem with piecewise constant control but without the assumption of a k-stage cost functional has been studied and many standard results on this subject are available [6], [30], [31]. A standard approach to this problem is to reformulate it as a discrete-time optimal control problem and apply the methods of mathematical programming. In discretizing the continuous system, an assumption is usually made that each subinterval is sufficiently short so that the first-order approximation for an integration on each subinterval is satisfactory. In the plasma heating problem, the assumption of small subintervals would lead to a very rapidly changing neutral beam current which is undesirable from the engineering standpoint. Therefore, we do not adopt this assumption. In fact, we shall not use any approximations in the derivation of necessary conditions for optimality. Although the approach without the small subinterval approximation is mentioned in many texts, for example [6], [36], among the references available to the author, none have given an explicit optimality condition for such a case.

1.3. Outline of Dissertation

In Part 1, consisting of Chapters II, III and IV, a mathematical discussion on three multiple stage optimal control problems is presented.

The two-stage optimal control problem is formulated in Section 2.1 and the existence of an optimal control pair is discussed in Section 2.2. In Sections 2.3 and 2.4 two sets of necessary conditions for an optimal control pair are derived via two methods. As a special case, a linear regulator problem with a two-stage quadratic cost is considered in Section 2.5. A sufficient condition for nonexistence of an optimal intermediate switching is also discussed for this problem.

In Chapter III, the two-stage problem is generalized to a multi-stage problem. The formulation of this problem is given in Section 3.1. Subsequently it is reduced to a standard optimal control problem by introducing a set of auxiliary controls. The existence question is discussed in Section 3.3. The possiblity of chattering controls is also discussed in this section. A numerical example is provided to illustrate the case when a chattering control is encountered.

The fixed k-stage optimal control problem is presented in Chapter IV. The basic formulation and an equivalent parameter optimization problem are given in Sections 4.1 and 4.2 respectively. A necessary condition in integral form is presented in Section 4.3.

Part 2 is composed of Chapters V and VI. The results derived in Part 1 are utilized in analyzing several problems of plasma heating by means of neutral injection. Chapter V is devoted to a discussion of the minimum input-energy plasma heating problem using a two-temperature model of the plasma which is given in Section 5.1. The existence and

the characterization of an optimal neutral injection heating program are discussed in Sections 5.2 and 5.3 respectively. The discussion of the optimal piecewise constant neutral injection program is then presented in Section 5.4.

A simplified single-temperature model of the plasma is adopted in Chapter VI. The minimum injection energy problem using this model is formulated in Section 6.1, where the existence of an optimal injection program is also discussed. The characterization of the optimal injection program is given in detail in Section 6.2. We then discuss the dynamics and stability of various ion temperature feedback control systems. The analysis includes the effect of measurement time-delay on the stability of the total system.

CHAPTER II

TWO-STAGE OPTIMAL CONTROL PROBLEM

The two-stage optimal control problem is introduced in this chapter. The existence and characterization of an optimal pair (control, switching time) are discussed.

2.1. Formulation of Two-Stage Optimal Control Problem

In this section we formulate an optimal control problem with a two-stage cost functional.

Consider a system described by a vector differential equation on a fixed time interval $T \triangleq [t_0, t_f]$

$$\dot{x} \triangleq \frac{dx}{dt} = f(x,u,t), \tag{1.1}$$

where $x(t) \in E^n$ and $u(t) \in E^m$ are the state and control vectors respectively. f is an n-dimensional vector-valued function which is assumed to be continuous in $E^n \times E^m \times T$. From now on, we denote "d/dt" by "$\cdot$". The initial and final conditions are given by

$$x(t_0) \in X_0, \qquad x(t_f) \in X_f, \tag{1.2}$$

where X_0 and X_f are nonempty closed sets in E^n.

Definition 1.1. A control u is said to be t_1-admissible for a given switching time $t_1 \in T$ if

1. there exists a corresponding unique solution of (1.1) which satisfies (1.2),

2. $u(\cdot)$ is measurable on T and $u(t)$ satisfies a control constraint

$$u(t) \in \begin{cases} \Omega_1, & \text{a.e. on } [t_0,t_1) \\ \Omega_2, & \text{a.e. on } [t_1,t_f] \end{cases}, \tag{1.3}$$

where the nonempty compact sets $\Omega_i \subseteq E^m$, $i = 1,2$ are called the control constraint sets. ‖

The set of all t_1-admissible controls is denoted by Δ_{t_1}. A pair (u,t_1) is called an <u>admissible control pair</u> if $t_1 \in T$ and $u \in \Delta_{t_1}$. We denote the set of all admissible control pairs by Λ.

<u>Definition 1.2.</u> For an unspecified switching time $t_1 \in T$, we define the two-stage cost functional $J(u,t_1)$ by

$$J(u,t_1) = \int_{t_0}^{t_1} L_1(x,u,t)dt + \int_{t_1}^{t_f} L_2(x,u,t)dt, \tag{1.4}$$

where x is a solution of (1.1) and (1.2), and L_i, $i = 1,2$ are continuous in $E^n \times E^m \times T$. ‖

Now the two-stage optimal control problem is defined as follows.

<u>Problem (T)</u>: Given a system (1.1), a control time duration T, initial and final conditions (1.2), control constraint sets Ω_1 and Ω_2 and a two-stage cost functional (1.4). Find an admissible control pair (u^*,t_1^*) which minimizes the two-stage cost $J(u,t_1)$, i.e.,

$$J(u^*,t_1^*) \leq J(u,t_1), \tag{1.5}$$

for any admissible control pair $(u,t_1) \in \Lambda$. ‖

A pair (u^*, t_1^*) is called an optimal control pair and its corresponding solution of (1.1) is called an optimal trajectory and denoted by x^*. A trajectory x corresponding to an admissible control pair (u, t_1) is called an admissible trajectory.

2.2. Existence of Optimal Control Pair

We give a set of conditions which guarantee the existence of an optimal control pair (u^*, t_1^*) by analyzing the properties of the attainable set at time t_f, viz, the set of all right hand end-points of the admissible trajectories $x(t)$ at $t = t_f$.

Consider the augmented systems described by

$$\dot{y}_i = \begin{bmatrix} L_i(x,u,t) \\ f(x,u,t) \end{bmatrix} \triangleq f_i(y_i, u, t), \quad i = 1,2, \tag{2.1}$$

where $y_i(t) \triangleq (x_i^0(t), x(t)) \in \mathbb{E}^{n+1}$, $i = 1,2$ will be referred to as augmented states. The first coordinate $x_i^0(t)$ represents the time evolution of the cost

$$x_i^0(t) = \int_{t_0}^{t} L_i(x(s), u(s), s)ds, \tag{2.2}$$

associated with the trajectory x specified by the last n coordinates of y_i, $i = 1,2$. We define augmented attainable sets $K_i(t;\tau,E)$, $i = 1,2$ by

$$K_i(t;\tau,E) \triangleq \bigcup_{y \in E} K_i(t;\tau,y), \tag{2.3}$$

where

$$K_i(t;\tau,y) \stackrel{\Delta}{=} \{y_i(t) : y_i(t) = y + \int_\tau^t f_i(y_i(s),u(s),s)ds,$$

$$u(s) \in \Omega_i, \text{ a.e. on } [\tau,t]\}. \quad (2.4)$$

Then the set of all end points at $t = t_f$ of the augmented trajectories corresponding to t_1-admissible controls, denoted by $\mathcal{K}(t_1)$, can be expressed as

$$\mathcal{K}(t_1) = K_2(t_f;t_1,K_1(t_1;t_0,y_0)), \quad (2.5)$$

where $y_0 \stackrel{\Delta}{=} [0,x_0]^T$ is an augmented initial point such that

$$y_0 \in Y_0 \stackrel{\Delta}{=} \{[0,x_0]^T : x_0 \in X_0\}. \quad (2.6)$$

We first prove the following theorem.

Theorem 2.1. Assume that

1. Ω_1,Ω_2 and X_0 are compact;
2. there exist nonempty sets $\mathcal{R}_1 \subseteq \mathbb{E}^n$ and $\mathcal{R}_2 \subseteq \mathbb{E}^n$ such that for $i = 1,2$, there exists a unique solution φ_i for (1.1) with $\varphi_i(t_0) = x_i$, $x_i \in \mathcal{R}_i$, for any control u satisfying $u(t) \in \Omega_i$, a.e. on T, and $\mathcal{R} \stackrel{\Delta}{=} \mathcal{R}_1 \cap \mathcal{R}_2$ is nonempty;
3. f is continuous and $\|f\|_n$ is bounded by $M_f < \infty$ in $\mathcal{R} \times \{\Omega_1 \cup \Omega_2\} \times T$, and L_i are continuous and $|L_i|$ are bounded by $M_\ell < \infty$ in $\mathcal{R} \times \Omega_i \times T$, $i = 1,2$;
4. $X_0 \subset \mathcal{R}$ and any trajectory φ with $\varphi(t_0) \in X_0$ corresponding to a control satisfying condition (2) of Definition 1.1 for some $t_1 \in T$ remains in $\mathcal{R}$, i.e.

$$\varphi(t) \in \mathcal{R} \ \forall\ t \in T; \quad (2.7)$$

5. the sets of augmented velocity vectors defined by

$$V_i(x,t) \triangleq \left\{ \begin{bmatrix} L_i(x,u,t) \\ f(x,u,t) \end{bmatrix} \in \mathbb{E}^{n+1} : u \in \Omega_i \right\}, \; i = 1,2, \quad (2.8)$$

are convex for fixed x and t in $\mathcal{R} \times T$.

Then $\mathcal{K}(t_1)$ is compact for any $t_1 \in T$ and varies continuously in t_1 with respect to the Hausdorff metric.+ ||

Proof: From conditions (3) and (4), for any solution y_1 of (2.1) with any control u satisfying $u(\tau) \in \Omega_1$ a.e. on $[t_0,t_1]$ and any $y_1(t_0) \in Y_0 = \{(0,x_0) : x_0 \in X_0\}$,

$$\|y_1(t)\|_{n+1} \leq \|y_1(t_0)\|_{n+1} + \int_{t_0}^{t} \|f_1(y_1(\tau),u(\tau),\tau)\|_{n+1}\, d\tau$$

$$\leq C_0 + 2M(t_f - t_0) < \infty, \quad (2.9)$$

where $C_0 \triangleq \max_{x_0 \in X_0} \|x_0\|_n$ and $M = \max\{M_f, M_\ell\}$. This implies that there exists a compact set $\overline{\mathcal{R}}_1 \subseteq \mathcal{R}$ such that $y_1(t) \in \overline{\mathcal{R}}_1$ for all $t \in T$. Hence under the conditions of this theorem, $K_1(t;t_0,x_0)$ is

+The Hausdorff metric $h(A,B)$ defined on the space of compact sets in $\mathbb{E}^r$ is given by

$$h(A,B) = \max\{\max_{a \in A} d(a,B), \max_{b \in B} d(b,A)\},$$

where $d(a,B)$ is the distance between the point a and the set B given by

$$d(a,B) = \min_{b \in B} \|a - b\|_r .$$

$\|\cdot\|_r$ is any valid norm defined on $\mathbb{E}^r$.

compact for each $t \in T$ and varies continuously in t [3]. Similarly for any solution y_2 of (2.1) with any control u satisfying $u(s) \in \Omega_2$ a.e. on $[\tau,t]$ and any $y_2(\tau) \in K_1(\tau;t_0,x_0)$,

$$\|y_2(t)\|_{n+1} \leq \|y_2(\tau)\|_{n+1} + \int_\tau^t \|f_2(y_2(s),u(s),s)\|_{n+1}\, ds$$

$$\leq \{C_0 + 2M(t_f - t_0)\} + 2M(t_f - t_0) < \infty. \tag{2.10}$$

Therefore $K_2(t;\tau,K_1(\tau;t_0,x_0))$ is also compact for each $t \in T$, $\tau \in T$, $\tau \leq t$, and varies continuously in t. Hence $\mathcal{K}(t_1) = K_2(t_f;t_1,K_1(t_1;t_0,y_0))$ is compact for any $t_1 \in T$. Next we show the continuity of $\mathcal{K}(t_1)$ with respect to t_1. Let $t_1 \leq t_2$, $t_i \in T$, $i = 1,2$. Suppose that a point P_1 is in $\mathcal{K}(t_1)$. Then there exist a point $P_3 \in K_1(t_1;t_0,y_0)$, a control u_1 satisfying $u_1(t) \in \Omega_2$ a.e. on $[t_1,t_f]$ and a corresponding augmented trajectory y_1 such that

$$P_1 = y_1(t_f) = P_3 + \int_{t_1}^{t_f} f_2(y_1(t),u_1(t),t)dt. \tag{2.11}$$

On the other hand, since $K_1(t;t_0,y_0)$ varies continuously in t, there exists a point $P_4 \in K_1(t_2;t_0,y_0)$ such that for every $\varepsilon_1 > 0$, there exists a $\delta_1 > 0$

$$|t_1 - t_2| < \delta_1 \Rightarrow \|P_3 - P_4\|_{n+1} < \varepsilon_1.$$

Let $P_2 \in \mathcal{K}(t_2) = K_2(t_f;t_2,K_1(t_2;t_0,y_0))$ be given by

$$P_2 = y_2(t_f) = P_4 + \int_{t_2}^{t_f} f_2(y_2(t),u_1(t),t)dt. \tag{2.12}$$

Now, under the assumptions of this theorem, it is known that the solutions for the augmented system (2.1) depend continuously on initial conditions [7]. In fact we have for every $\varepsilon > 0$, there exists a $\delta_2 > 0$ such that

$$|t_1 - t_2| + \|P_3 - P_4\|_{n+1} < \delta_2 \Rightarrow \|y_1(t) - y_2(t)\|_{n+1} < \varepsilon, \tag{2.13}$$

uniformly on $[t_2, t_f]$. Hence for a $\delta_1 > 0$ such that $\varepsilon_1 + \delta_1 < \delta_2$, we have for every $\varepsilon > 0$

$$|t_1 - t_2| < \delta_1 \Rightarrow \|P_1 - P_2\|_{n+1} = \|y_1(t_f) - y_2(t_f)\|_{n+1} < \varepsilon. \tag{2.14}$$

This implies that $K(t_1)$ varies continuously in t_1 on T. $\|$

Next, we find a subset of $K(t_1)$ which corresponds to a set of end points of admissible trajectories. Since the end point of any admissible trajectory must lie in X_f, the end point of any augmented admissible trajectory must be in $K(t_1) \cap Y_f$ for some $t_1 \in T$ where

$$Y_f \triangleq \{y_f = (x_f^0, x_f) \in \mathbb{E}^{n+1} : x_f^0 \text{ free, } x_f \in X_f\}. \tag{2.15}$$

In other words, any pair of $t_1 \in T$ and the end point y_f of an augmented trajectory corresponding to some t_1-admissible control, $(t_1, y_f) \in \mathbb{E}^{n+2}$ lies in $A \triangleq G \cap Z_f$ where,

$$G \triangleq \bigcup_{t_1 \in T} \{(t_1, y) \in \mathbb{E}^{n+2} : y \in K(t_1)\}, \tag{2.16}$$

$$Z_f \triangleq \bigcup_{t_1 \in T} \{(t_1, y) \in \mathbb{E}^{n+2} : y = (x^0, x) \in Y_f\}. \tag{2.17}$$

The set G is illustrated in Fig. 2.1. Suppose the set A is nonempty and compact, then an optimal control pair (u^*, t_1^*) exists, since for a compact A, there exists a point a^* in A which has the smallest second coordinate value among all points in A. The first element of a^* is the optimal switching time t_1^*. The optimal control u^* is given by the control which realizes the corresponding augmented trajectory y^* whose end point $y^*(t_f)$ coincides with the last $n+1$ elements of a^*.

We shall show the compactness of $A = G \cap Z_f$, hence the existence of an optimal control pair (u^*, t_1^*), in the following theorem.

<u>Theorem 2.2</u>. In addition to the assumptions (1)-(5) of Theorem 2.1, we assume that

6. the set of t_1-admissible controls Δ_{t_1} is nonempty for t_1 in a nonempty closed subset $T_s \subseteq T$.

Then $A = G \cap Z_f$ is a nonempty compact set and an optimal control pair (u^*, t_1^*) exists. ||

<u>Proof</u>: The nonemptiness of $G \cap Z_f$ is guaranteed by assumption (6). Hence it remains to show the compactness of $G \cap Z_f$. Since X_f was assumed to be closed in $\mathbb{E}^n$, Z_f is also closed in $\mathbb{E}^{n+2}$. Therefore if G is compact then $G \cap Z_f$ is also compact. Obviously G is bounded; we only need to show the closedness of G. Let $\{g_i\}$ be a sequence of points of G which converges to a point $\bar{g}$. We show that $\bar{g} \in G$ by contradiction.

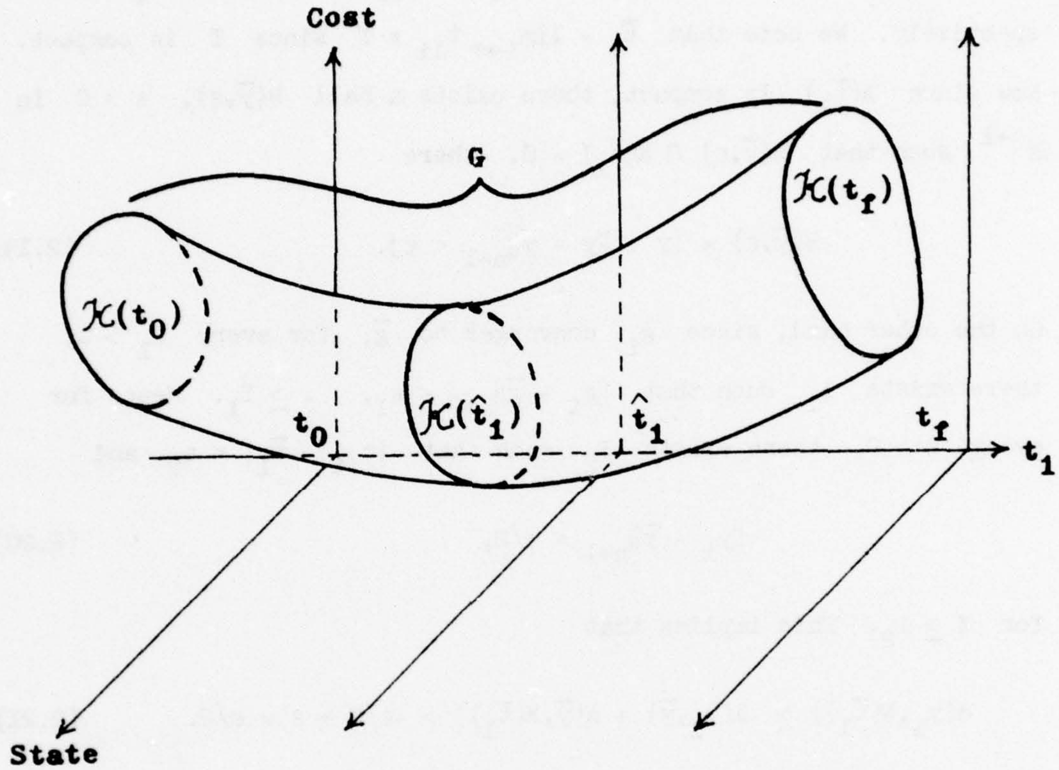

Fig. 2.1. Set $\mathcal{K}(t_1)$ and its Graph G

Suppose $\bar{g} \notin G$. Let $g_i \in \mathbb{E}^{n+2}$ and $\bar{g} \in \mathbb{E}^{n+2}$ be given by

$$g_i \triangleq \begin{bmatrix} t_{1i} \\ y_i \end{bmatrix}, \quad \bar{g} = \begin{bmatrix} \bar{t}_1 \\ \bar{y} \end{bmatrix} . \tag{2.18}$$

Then $g_i \in G$ and $\bar{g} \notin G$ implies $y_i \in \mathcal{K}(t_{1i})$ and $\bar{y} \notin \mathcal{K}(\bar{t}_1)$ respectively. We note that $\bar{t}_1 = \lim_{i\to\infty} t_{1i} \in T$ since T is compact. Now since $\mathcal{K}(\bar{t}_1)$ is compact, there exists a ball $B(\bar{y},\varepsilon)$, $\varepsilon > 0$ in $\mathbb{E}^{n+1}$ such that $B(\bar{y},\varepsilon) \cap \mathcal{K}(\bar{t}_1) = \emptyset$, where

$$B(\bar{y},\varepsilon) = \{y : \|y - \bar{y}\|_{n+1} < \varepsilon\}. \tag{2.19}$$

On the other hand, since g_i converges to $\bar{g}$, for every $\varepsilon_1 > 0$, there exists I_1 such that $\|g_i - \bar{g}\|_{n+2} < \varepsilon_1$, $i \geq I_1$. Hence for every $\varepsilon > 0$, there exists I_2 such that $|t_{1i} - \bar{t}_1| < \varepsilon_2$ and

$$\|y_i - \bar{y}\|_{n+1} < \varepsilon/2, \tag{2.20}$$

for $i \geq I_2$. This implies that

$$d(y_i,\mathcal{K}(\bar{t}_1)) \geq -d(y_i,\bar{y}) + d(\bar{y},\mathcal{K}(\bar{t}_1)) > -\varepsilon/2 + \varepsilon = \varepsilon/2. \tag{2.21}$$

Hence $h(\mathcal{K}(t_i),\mathcal{K}(\bar{t}_1)) > \varepsilon/2$ for some $\varepsilon > 0$. This contradicts the continuity of $\mathcal{K}(t_1)$ with respect to the Hausdorff metric. Therefore $\bar{y} \in \mathcal{K}(\bar{t}_1)$, i.e., $\bar{g} \in G$. Thus G is compact and an optimal control pair (u^*,t_1^*) exists. ‖

<u>Remark</u>: Let $G_s \subset \mathbb{E}^{n+2}$ be defined by

$$G_s \triangleq \bigcup_{t_1 \in T_s} \{(t_1,y) \in \mathbb{E}^{n+2} : y \in \mathcal{K}(t_1)\}. \tag{2.22}$$

Then assumption (6) implies

$$G \cap Z_f = G_s \cap Z_f \neq \emptyset. \tag{2.23}$$

Hence t_1^* exists in T_s. ‖

We note that the conditions (1)-(6) are similar to those of the standard existence theorem [3]. In fact, under these conditions, there exist optimal controls u_i^*, $i = 1,2$ for the problems with the cost functional

$$J_i(u) = \int_{t_0}^{t_f} L_i(x,u,t)dt. \tag{2.24}$$

(See Appendix A.)

2.3. Necessary Condition for Optimality

Under a set of appropriate assumptions, a necessary condition for an optimal control pair (u^*,t_1^*) can be derived via direct computation of first order variations. The result is a modification of a well-known necessary condition in calculus of variations for a standard problem with no intermediate switching. A set of assumptions and conditions are summarized in the following theorem.

Theorem 3.1. Consider system (1.1) and cost (1.4). Assume that the control time interval $T = [t_0,t_f]$ is fixed. Let the initial point $x(t_0) = x_0$ be fixed and the terminal point $x(t_f)$ be free. Assume that u is continuous everywhere except possibly at t_1. Also assume that $u(t)$ is unconstrained. Let f, L_1 and L_2 be twice continuously differentiable in x and u, and continuously differentiable in t.

Define the Hamiltonians $H_i(x,p,u,t)$ by

$$H_i(x,p,u,t) = L_i(x,u,t) + p^T f(x,u,t), \qquad i = 1,2, \tag{3.1}$$

where $p(t)$ is an n-dimensional adjoint vector. We use the superscript "*" to denote optimal quantities. Now, assume there exists an optimal control pair (u^*,t_1^*) where $t_1^* \in (t_0,t_f)$. Then it is necessary that there exists a function p^* such that x^* and p^* satisfy a set of canonical equations given by

$$\dot{x}^* = f(x^*,u^*,t) = \begin{Bmatrix} \partial H_1/\partial p\ (x^*,p^*,u_1^*,t), & t_0 \le t < t_1 \\ \partial H_2/\partial p\ (x^*,p^*,u_2^*,t), & t_1 < t \le t_f \end{Bmatrix}, \tag{3.2}$$

$$\dot{p}^* = \begin{Bmatrix} -\partial H_1/\partial x\ (x^*,p^*,u_1^*,t), & t_0 \le t < t_1 \\ -\partial H_2/\partial x\ (x^*,p^*,u_2^*,t), & t_1 < t \le t_f \end{Bmatrix}, \tag{3.3}$$

with boundary conditions

$$x^*(t_0) = x_0, \qquad p^*(t_f) = 0, \tag{3.4}$$

$$p^*(t_1^*-) = p^*(t_1^*+), \tag{3.5}$$

where $p^*(t_1^*-)$ and $p^*(t_1^*+)$ are the left and right limits of $p^*(t)$ at t_1^* respectively and u_i^*, $i = 1,2$ are solutions of

$$\begin{Bmatrix} \partial H_1/\partial u\ (x^*(t),p^*(t),u_1^*(t),t) = 0, & t_0 \le t < t_1^* \\ \partial H_2/\partial u\ (x^*(t),p^*(t),u_2^*(t),t) = 0, & t_1^* < t \le t_f \end{Bmatrix}. \tag{3.6}$$

Also the following transversality condition is satisfied at t_1^*

$$H_1(x^*(t_1^*),p^*(t_1^*),u_1^*(t_1^*),t_1^*) = H_2(x^*(t_1^*),p^*(t_1^*),u_2^*(t_1^*),t_1^*), \quad (3.7)$$

where we have used (3.5). ||

Remarks:

(R.1) When $t_1^* = t_0$, the boundary condition (3.5) is irrelevant and the transversality condition (3.7) is modified to

$$H_1(x_0,p^*(t_0),u_1^*(t_0),t_0) \leq H_2(x_0,p^*(t_0),u_2^*(t_0),t_0). \quad (3.8)$$

Similarly, for $t_1^* = t_f$, (3.5) is again irrelevant and (3.7) is now replaced by

$$H_1(x^*(t_f),0\ ,u_1^*(t_f),t_f) \geq H_2(x^*(t_f),0,u_2^*(t_f),t_f), \quad (3.9)$$

or

$$L_1(x^*(t_f),u_1^*(t_f),t_f) \leq L_2(x^*(t_f),u_2^*(t_f),t_f). \quad (3.10)$$

(R.2) The conditions (3.6) and (3.7) are the two extra conditions for our two-stage aproblem. (3.6) states the continuity of $p^*(t)$ at t_1^* and (3.7) states the matching of H_1^* and H_2^* at t_1^*. ||

A lengthy but straightforward derivation of the foregoing conditions is given in Appendix B.

2.4. Decomposition into Standard Problems

We derive a necessary condition for an optimal control pair (u^*,t_1^*) in the form of a maximum principle by decomposing the original problem into two standard problems. In this section, f, L_1 and L_2

are assumed to be continuously differentiable in x, u and t. We start our discussion from the case $t_1^* \in (t_0, t_f)$.

First we consider an auxiliary problem which corresponds to the second stage of the original problem.

<u>Problem 1</u>. Given a system $\dot{x} = f(x,u,t)$, a control time interval $[t_1, t_f]$, initial and final conditions

$$x(t_1) = x_1, \qquad x(t_f) \in X_f, \tag{4.1}$$

and a control constraint that $u(\cdot)$ is measurable on $[t_1, t_f]$ and

$$u(t) \in \Omega_2, \quad \text{a.e. on } [t_1, t_f]. \tag{4.2}$$

Find an optimal control u^* which is admissible and minimizes the cost $J_2(u)$ given by

$$J_2(u) = \int_{t_1}^{t_f} L_2(x,u,t)dt, \tag{4.3}$$

over all admissible controls. ||

<u>Note</u>: In this section, a control u is said to be <u>admissible</u> whenever

1. there exists a corresponding solution for the system equation which satisfies the initial and final conditions,
2. u satisfies a control constraint.

It should be noted that the system equation, initial and final conditions and control constraint may vary from problem to problem. ||

Problem 1 is of standard form and can be solved using the Pontryagin maximum principle. Let the Hamiltonian H_2 be defined by

$$H_2(x,p,u,t) = -L_2(x,u,t) + p_2^T f(x,u,t), \tag{4.4}$$

where p_2 is an adjoint vector. Let u_2^* and x^* be an optimal control and its corresponding trajectory. Then there exists (see the maximum principle presented in Appendix C) a corresponding adjoint vector p^* such that

a) $$\dot{x}^* = \{\partial H_2/\partial p_2\,(x^*,p_2^*,u_2^*,t)\}^T, \quad x^*(t) = x_1,\ x^*(t_f) \in X_f, \tag{4.5}$$

$$\dot{p}_2^* = -\{\partial H_2/\partial x\,(x^*,p_2^*,u_2^*,t)\}^T, \quad p_2^*(t_f) \perp \Pi_f^*\ ; \tag{4.6}$$

b) $$\max_{u_2 \in \Omega_2} H_2(x^*(t),p_2^*(t),u_2,t) = H_2(x^*(t),p_2^*(t),u_2^*(t),t)$$

$$\text{a.e. on } [t_1,t_f], \tag{4.7}$$

where Π_f^* is a tangent plane to x_f at $x^*(t_f)$.

Now, suppose that we solve this problem for fixed t_1 and x_1 and express the optimal cost J_2^* as a function of t_1 and x_1, i.e.,

$$J_2^* = J_2^*(x_1,t_1). \tag{4.8}$$

Then we can reduce the original two-stage cost (1.4) to a cost in standard form by substituting J_2^*. Thus the original problem can be reformulated as follows:

<u>Problem 2</u>. Given a system $\dot{x} = f(x,u,t)$, initial and final conditions

$$x(t_0) \in X_0, \quad t_1 \in (t_0,t_f) \text{ and } x(t_1) \text{ free}, \tag{4.9}$$

and a control constraint that $u(\cdot)$ is measurable on $[t_0,t_1]$ and

$$u(t) \in \Omega_1, \quad \text{a.e. on } [t_0,t_1], \tag{4.10}$$

where $[t_0,t_1]$ is the domain of definition of u. Find an optimal control u^* defined on an optimal interval $[t_0,t_1^*]$, $t_1^* \in (t_0,t_f)$ which minimizes the cost

$$\tilde{J}(u) = \int_{t_0}^{t_1} L_1(x,u,t)dt + J_2^*(x(t_1),t_1), \tag{4.11}$$

ie., $\tilde{J}(u^*) \leq \tilde{J}(u)$ for any admissible control u defined on any interval $[t_0,t_1]$ where $t_1 \in (t_0,t_f)$. ||

Remark: Since we have assumed that $t_1^* \in (t_0,t_f)$, the constraint $t_1 \in (t_0,t_f)$ is irrelevant. Therefore this problem reduces to a problem with free terminal time and end point. The other cases $t_1^* = t_0$ and $t_1^* = t_f$ will be treated later. ||

Using the results in Remark 3 of the maximum principle in Appendix C, we have the following set of necessary conditions for an optimal control u^* and an optimal terminal time t_1^*. Let the Hamiltonian H_1 be defined as

$$H_1(x,p,u,t) = -L_1(x,u,t) + p^T f(x,u,t). \tag{4.12}$$

Then,

a) there exists an adjoint vector p_1^* defined on $[t_0,t_1^*]$ such that

$$\dot{x}^* = \{\partial H_1/\partial p_1\ (x^*, p_1^*, u_1^*, t)\}^T, \qquad x^*(t_0) \in X_0, \tag{4.13}$$

$$\dot{p}_1^* = -\{\partial H_1/\partial x\ (x^*, p_1^*, u_1^*, t)\}^T, \qquad p_1^*(t_0) \perp \Pi_0^*, \tag{4.14}$$

where Π_0^* is a tangent plane to X_0 at $x^*(t_0)$;

b) $\max\limits_{u_1 \in \Omega_1} H_1(x^*(t), p_1^*(t), u_1, t) = H_1(x^*(t), p_1^*(t), u_1^*(t), t)$

a.e. on $[t_0, t_1^*]$; (4.15)

c) $p_1^*(t_1^*)$ and $H_1^*|_{t_1^*}$ satisfy the following transversality conditions,

$$p_1^*(t_1^*) = -\{\partial J_2^*/\partial x\ (x^*(t_1^*), t_1^*)\}^T, \tag{4.16}$$

$$H_1(x^*(t_1^*), p_1^*(t_1^*), u_1^*(t_1^*), t_1^*) = \partial J_2^*/\partial t_1\ (x^*(t_1^*), t_1^*). \tag{4.17}$$

On the other hand, by the help of additional regularity assumptions, the right hand sides of (4.16) and (4.17) can be computed as

$$\partial J_2^*/\partial x_1(x_1, t_1) = -p_2^{*T}(t_1), \tag{4.18}$$

$$\begin{aligned} \partial J_2^*/\partial t_1(x_1, t_1) &= dJ_2^*/dt_1(x_1, t_1) - (\partial J_2^*/\partial x_1(x_1, t_1))dx_1/dt_1 \\ &= H_2(x^*(t_1), p_2^*(t_1), u_2^*(t_1), t_1). \end{aligned} \tag{4.19}$$

By comparing (4.16) to (4.18) and (4.17) to (4.19) we have

$$p_1^*(t_1^*) = p_2^*(t_1^*), \tag{4.20}$$

$$H_1^*|_{t_1^*} = H_2^*|_{t_1^*}, \tag{4.21}$$

where we abbreviate $H_i(x^*, p_i^*, u_i^*, t)$ by H_i^*, $i = 1,2$.

Thus we have established the following theorem.

Theorem 4.1. Suppose that (u^*,t_1^*), $t_1^* \in (t_0,t_f)$ is an optimal control pair for the original two-stage cost problem, then it is necessary that there exists an adjoint vector p^* such that,

a) an optimal trajectory x^* and p^* satisfy the set of canonical equations given by

$$\left.\begin{array}{l} \dot{x}^* = \begin{cases} \{\partial H_1^*/\partial p\}^T, & t_0 \le t < t_1^*, \\ \{\partial H_2^*/\partial p\}^T, & t_1^* < t \le t_f, \end{cases} \\ \\ x^*(t_0) \in X_0 \quad \text{and} \quad x^*(t_f) \in X_f \end{array}\right\}; \tag{4.22}$$

$$\left.\begin{array}{l} \dot{p}^* = \begin{cases} -\{\partial H_1^*/\partial x\}^T, & t_0 \le t < t_1^*, \\ -\{\partial H_2^*/\partial x\}^T, & t_1^* < t \le t_f, \end{cases} \\ \\ p^*(t_1^*-) = p^*(t_1^*+), \\ \\ p^*(t_0) \perp \Pi_0^* \quad \text{and} \quad p^*(t_f) \perp \Pi_f^* \end{array}\right\}; \tag{4.23}$$

b) the Hamiltonians H_i, $i = 1,2$ are maximized by u^* as

$$\left.\begin{array}{l} \max\limits_{u\in\Omega_1} H_1(x^*(t),p^*(t),u,t) = H_1(x^*(t),p^*(t),u^*(t),t), \\ \qquad\qquad\qquad\qquad \text{a.e.} \ [t_0,t_1^*) \\ \\ \max\limits_{u\in\Omega_1} H_2(x^*(t),p^*(t),u,t) = H_2(x^*(t),p^*(t),u^*(t),t), \\ \qquad\qquad\qquad\qquad \text{a.e.} \ (t_1^*,t_f] \end{array}\right\}; \tag{4.24}$$

c) The following transversality condition is satisfied at t_1^*,

$$H_1^*|_{t_1^*} = H_2^*|_{t_1^*} \,. \qquad \| \tag{4.25}$$

Now, for the cases $t_1^* = t_0$ and $t_1^* = t_f$, we have the following result:

<u>Theorem 4.2.</u> Suppose that $t_1^* = t_0$ (resp. $t_1^* = t_f$), then it is necessary that there exists an adjoint vector p^* such that for all $t \in T$,

a)
$$\left.\begin{array}{l} \dot{x}^* = \{\partial H_2^*/\partial p\}^T, \quad (\text{resp. } \dot{x}^* = \{\partial H_1^*/\partial p\}^T) \\ x^*(t_0) \in X_0 \text{ and } x^*(t_f) \in X_f \end{array}\right\}, \tag{4.26}$$

$$\left.\begin{array}{l} \dot{p}^* = -\{\partial H_2^*/\partial x\}^T, \quad (\text{resp. } \dot{p}^* = -\{\partial H_1^*/\partial x\}^T) \\ p^*(t_0) \perp \Pi_0^* \text{ and } p^*(t_f) \perp \Pi_f^* \end{array}\right\}; \tag{4.27}$$

b) u^* satisfies

$$\left.\begin{array}{l} \max\limits_{u\in\Omega_2} H_2(x^*(t),p^*(t),u,t) = H_2(x^*(t),p^*(t),u^*(t),t) \\ (\text{resp. } \max\limits_{u\in\Omega_1} H_1(x^*(t),p^*(t),u,t) = H_1(x^*(t),p^*(t),u^*(t),t)) \\ \qquad\qquad \text{a.e. on } T \end{array}\right\}; \tag{4.28}$$

c) at $t_1^* = t_0$ (resp. $t_1^* = t_f$) we have

$$\left.\begin{aligned} &H_1(x^*(t_0),p^*(t_0),u_1^*(t_0),t_0) \le H_2(x^*(t_0),p^*(t_0),u_2^*(t_0),t_0) \\ &(\text{resp. } H_1(x^*(t_f),p^*(t_f),u_1^*(t_f),t_f) \ge H_2(x^*(t_f),p^*(t_f),u_2^*(t_f),t_f)) \end{aligned}\right\}, \tag{4.29}$$

where u_i^* maximizes H_i^*. ||

Proof: We will consider the case $t_1^* = t_0$ only. The other case $t_1^* = t_f$ can be treated similarly. The conditions (4.26), (4.27), and (4.28) are derived respectively from the conditions (4.23), (4.24), and (4.25) of Theorem 4.1. Now, from (4.11) we have for $t_1 \in T$,

$$\begin{aligned} J^*(t_1) \overset{\Delta}{=} \tilde{J}(u^*) &= \int_{t_0}^{t_1} L(x^*, u^*, t)dt + J_2^*(x^*(t_1),t_1) \\ &= \int_{t_0}^{t_1} L_1^*|_t \, dt + J_2^*(x^*(t_1),t_1). \end{aligned} \tag{4.30}$$

Then

$$\begin{aligned} \frac{dJ^*}{dt_1}(t_1) &= \frac{d}{dt_1}\{ \int_{t_0}^{t_1} L_1^*|_t \, dt + J_2^*(x^*(t_1),t_1)\} \\ &= L_1^*|_{t_1} + \partial J_2^*/\partial x|_{t_1} \dot{x}^*(t_1) + \partial J_2^*/\partial t_1|_{t_1}. \end{aligned} \tag{4.31}$$

Now using (4.18) and (4.19)

$$\frac{dJ^*}{dt_1}(t_1) = L_1^*|_{t_1} - p^*(t_1)^T f^*|_{t_1} + H_2^*|_{t_1} = -H_1^*|_{t_1} + H_2^*|_{t_1}. \tag{4.32}$$

For the optimality of $t_1^* = t_0$, we must have

$$\frac{dJ^*}{dt_1}(t_0) \ge 0. \tag{4.33}$$

Therefore

$$-H_1^*|_{t_0} + H_2^*|_{t_0} \ge 0. \tag{4.34}$$

This gives (4.29). ||

Remark: When $H_1^*|_{t_1}$ and/or $H_2^*|_{t_1}$ do not exist (for example, if u^* is discontinuous at t_1), we replace $H_1^*|_{t_1}$ and $H_2^*|_{t_1}$ by the left and right limits $H_1^*|_{t_1-}$ and $H_2^*|_{t_1+}$ defined by

$$\left.\begin{aligned} H_1^*|_{t_1-} &= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} H_1^*|_{t_1-\varepsilon}\,, \\ H_2^*|_{t_1+} &= \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} H_2^*|_{t_1+\varepsilon}\,. \end{aligned}\right\} \tag{4.35}$$

||

2.5. Special Case

As a special case, we consider a state regulator problem with a linear system and a two-stage quadratic cost. The system and the cost are defined on $T = [t_0,t_f]$ by

$$\left.\begin{aligned} &\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t), \\ &\mathbf{x}(t_0) = \mathbf{x}_0 \neq 0 \text{ given and } \mathbf{x}(t_f) \text{ free,} \end{aligned}\right\} \tag{5.1}$$

$$J(u) = \int_{t_0}^{t_1} \frac{1}{2}(\mathbf{x}(t)^T Q_1 \mathbf{x}(t) + u(t)^T R_1 u(t))dt + \int_{t_1}^{t_f} \frac{1}{2}(\mathbf{x}(t)^T Q_2 \mathbf{x}(t) + u(t)^T R_2 u(t))dt, \tag{5.2}$$

where $A(n\times n)$, $B(n\times m)$, $Q_i(n\times n)$, $R_i(m\times m)$ are constant matrices and $Q_i^T = Q_i \geq 0$, $R_i^T = R_i > 0$ for $i = 1,2$. We consider no constraint on $u(t)$, i.e., $\Omega_1 = \Omega_2 = \mathbb{E}^m$.

For this problem, the cost for the corresponding decomposed problem (1) is given by

$$J_2 = \int_{t_1}^{t_f} \frac{1}{2}(\mathbf{x}(t)^T Q_2 \mathbf{x}(t) + u(t)^T R_2 u(t))dt. \tag{5.3}$$

It is well-known that the adjoint vector $p_2(t)$ and the optimal control $u_2^*(t)$ are given by

$$\left.\begin{aligned} p_2(t) &= -S_2(t)x^*(t), \\ u_2^*(t) &= R_2^{-1}B^T p_2(t) = -R_2^{-1}B^T S_2(t)x^*(t), \end{aligned}\right\} \tag{5.4}$$

where $n \times n$ matrix $S_2(t)$ is a positive definite solution of matrix Riccati equation:

$$\left.\begin{aligned} \dot{S}_2(t) &= -S_2(t)A - A^T S_2(t) - Q_2 + S_2(t)BR_2^{-1}B^T S_2(t), \\ S_2(t_f) &= 0. \end{aligned}\right\} \tag{5.5}$$

Also the optimal cost J_2^* can be written as

$$J_2^*(x(t_1),t_1) = \frac{1}{2}\, x(t_1)^T\, S_2(t_1)x(t_1). \tag{5.6}$$

Now, using (5.6), the cost for the decomposed problem (2) is given by,

$$\tilde{J} = \frac{1}{2}\, x(t_1)^T\, S_2(t_1)x(t_1) + \int_{t_0}^{t_1} \frac{1}{2}\,(x(t)^T Q_1 x(t) + u(t)^T R_1 u(t))dt. \tag{5.7}$$

For a free t_1 and free $x(t_1)$, problem (2) can be solved as follows:

$$\left.\begin{aligned} p_1(t) &= -\,S_1(t)x^*(t), \\ u_1^*(t) &= R_1^{-1}B^T p_1(t) = -R_1^{-1}B^T S_1(t)x^*(t), \end{aligned}\right\} \tag{5.8}$$

where $S_1(t)$ is an $n \times n$ positive definite matrix solution of another matrix Riccati equation:

$$
\left.\begin{aligned}
\dot{S}_1(t) &= -S_1(t)A - A^T S_1(t) - Q_1 + S_1(t)BR_1^{-1}B^T S_1(t), \\
S_1(t_1) &= S_2(t_1).
\end{aligned}\right\} \tag{5.9}
$$

Next we compute H_i^*, $i = 1,2$.

$$
H_i^* = -\frac{1}{2}(x_i^{*T} Q_i x_i^* + u_i^{*T} R_i u_i^*) + p_i^T (Ax_i^* + Bu_i^*), \tag{5.10}
$$

where x_i^*, $i = 1,2$ are solutions of (5.1) with controls (5.8) and (5.4) on $[t_0, t_1)$ and $[t_1, t_f]$ respectively. Algebraic computation using (5.4) and (5.8) yields,

$$
\left.\begin{aligned}
H_1^*|_{t_1^*} &= \frac{1}{2} x^*(t_1^*)^T \dot{S}_1(t_1^*) x^*(t_1^*), \\
H_2^*|_{t_1^*} &= \frac{1}{2} x^*(t_1^*)^T \dot{S}_2(t_1^*) x^*(t_1^*).
\end{aligned}\right\} \tag{5.11}
$$

Hence the transversality condition $H_1^*|_{t_1^*} = H_2^*|_{t_1^*}$ is

$$
\begin{aligned}
0 &= x^*(t_1^*)^T \{-\dot{S}_1(t_1^*) + \dot{S}_2(t_1^*)\} x^*(t_1^*) \\
&= x^*(t_1^*)^T \{Q_1 - Q_2 - S_2(t_1^*)B(R_1^{-1} - R_2^{-1})B^T S_2(t_1^*)\} x^*(t_1^*).
\end{aligned} \tag{5.12}
$$

We notice that if the matrix $\hat{Q}(t)$ defined by

$$
\hat{Q}(t) \triangleq Q_1 - Q_2 - S_2(t)B(R_1^{-1} - R_2^{-1})B^T S_2(t) \tag{5.13}
$$

is either positive or negative definite for all $t \in T$, then the transversality condition (5.12) can be satisfied only by $x^*(t_1^*) = 0$. But since the canonical equation for $x^*(t)$ on $[t_0, t_1^*]$ is given by

$$
\dot{x}^*(t) = (A - BR_1^{-1}B^T S_1(t)) x^*(t), \tag{5.14}
$$

then $x^*(t) \equiv 0$, if $x^*(t) = 0$ at some time. This contradicts the condition $x(t_0) \neq 0$. Hence the definiteness of the matrix $\hat{Q}(t)$ gives a criterion for the nonexistence of an intermediate switching. This is summarized in the next theorem.

Theorem 5.1. Suppose $x_0 \neq 0$ and the matrix $\hat{Q}(t)$ defined by (5.13) is positive (resp. negative) definite for all $t \in T$, then $t_1^* = t_0$ (resp. $t_1^* = t_f$). ||

Proof: We already know that t_1^* is one of the end points t_0 or t_f. Hence we only need to show that $t_1^* = t_0$ for the positive definite $\hat{Q}(t)$. The other case can be shown similarly. The positive definiteness of $\hat{Q}(t)$ and (5.12) gives

$$-H_1^*|_{t_1} + H_2^*|_{t_1} = \frac{1}{2} x^*(t_1)^T \hat{Q}(t_1) x^*(t_1) > 0, \qquad (5.15)$$

for all $t_1 \in T$. Now let $J^*(t_1) = \min_{u \in \Delta_{t_1}} J(u,t_1)$ as it was defined in Section 2.2. Then we have

$$\frac{dJ^*(t_1)}{dt_1} = -H_1^*|_{t_1} + H_2^*|_{t_1}, \qquad (5.16)$$

as is shown in Appendix B. Therefore $dJ^*(t_1)/dt_1 > 0$ for all $t_1 \in T$. This gives $t_1^* = t_0$. ||

Thus, for this special case we have derived a sufficient condition for t_1^* to be one of the end points t_0 or t_f.

CHAPTER III

MULTI-STAGE OPTIMAL CONTROL PROBLEM

The two-stage optimal control problem is generalized to the multi-stage optimal control problem in this chapter. It is shown that the multi-stage problem is reducible to a standard optimal control problem by introducing a set of auxiliary controls.

3.1. Formulation of Multi-Stage Optimal Control Problem

We will now generalize the two-stage optimal control problem to the multi-stage optimal control problem by allowing the integrand of the cost to switch any number of times. The integrand can switch from one form to another among N given forms.

As before, we let the system, the control time interval and the initial and final conditions be given as

$$\dot{x} = f(x,u,t), \tag{1.1}$$

$$t \in T \triangleq [t_0,t_f], \tag{1.2}$$

$$x(t_0) \in X_0 \subseteq \mathbb{E}^n, \qquad x(t_f) \in X_f \subseteq \mathbb{E}^n. \tag{1.3}$$

Let $\{\Omega_i, L_i(\cdot,\cdot,\cdot)\}$, $i = 1,2,\ldots,N$ be N given pairs of a control constraint set and cost integrand such that Ω_i is active when L_i is chosen as a current integrand of the cost functional. We assume Ω_i to be a convex compact set in $\mathbb{E}^m$ and, f and L_i to be continuous in $\mathbb{E}^n \times \mathbb{E}^m \times \mathbb{E}$, $i = 1,2,\ldots,N$.

Definition 1.1. We define a switching schedule $\mathcal{S}_K$ to be a $(2K-1)$-tuple $[t_1, t_2, \ldots, t_{K-1}; i_1, i_2, \ldots, i_K]$ where K is the number of stages; t_j, $j = 1,2,\ldots,K-1$, $t_0 < t_1 < t_2 < \ldots < t_{K-1} < t_f$, are the switching times; and i_j, $j = 1,2,\ldots,K$ are the indices of pairs (Ω_{i_j}, L_{i_j}) chosen for subintervals $[t_{j-1}, t_j)$, $j = 1,2,\ldots,K$, $t_K \triangleq t_f$. ‖

It should be noted that, since the number of stages is free, K may vary from one schedule to another.

Definition 1.2. A control u is said to be $\mathcal{S}_K$-admissible if

1. *there exists a corresponding solution of* (1.1) *which* satisfies (1.3);
2. $u(\cdot)$ is measurable on T and $u(t) \in \Omega_{i_j}$ a.e. on $[t_{j-1}, t_j)$, $j = 1,2,\ldots,K$. ‖

The performance of an $\mathcal{S}_K$-admissible control u is measured by a cost functional $J(u)$ given by

$$J(u) \triangleq \sum_{j=1}^{K} \int_{t_{j-1}}^{t_j} L_{i_j}(x(t),u(t),t)dt. \tag{1.4}$$

Now the multi-stage optimal control problem can be stated as follows:

Problem (M): Given a system (1.1), a control time interval (1.2), initial and final conditions (1.3) and N pairs of control constraint sets and cost integrands $\{\Omega_i, L_i\}$, $i = 1,2,\ldots,N$. Find a switching schedule $\mathcal{S}^*_{K^*} = [t^*_1, \ldots, t^*_{K^*-1}; i^*_1, \ldots, i^*_{K^*}]$ and $\mathcal{S}^*_{K^*}$-admissible control u^* such that

$$J(u^*) = \sum_{j=1}^{K^*} \int_{t_{j-1}}^{t_j^*} L_{i_j^*} (x^*,u^*,t)dt \leq J(u) \tag{1.5}$$

for any switching schedule S_K and for any S_K-admissible control u. ||

Such a pair $(S^*_{K^*},u^*)$ is called an optimal pair, i.e., the superscript "*" denotes optimality as before. A diagram showing the information flow in this problem is given in Figure 3.1.

The multi-stage problem differs significantly from the two-stage problem in two ways: (1) the sequence of integrands of the cost functional is not fixed, (2) the number of switchings is not fixed. In particular, we do not know beforehand whether the optimal solution for the multi-stage problem has a finite number of stages. This question involves the existence of an optimal auxiliary control which will be introduced in the next section. We will discuss this point in later sections.

3.2 Reduction to Standard Problem

In this section, we reduce the multi-stage optimal control problem to a standard optimal control problem using auxiliary controls.

First we define N auxiliary controls v_i, $i = 1,2,\dots,N$ which satisfy the constraint:

$$\left.\begin{aligned} &v_i(t) = 0 \text{ or } 1, \quad i = 1,2,\dots,N \\ &\sum_{i=1}^{N} v_i(t) = 1 \end{aligned}\right\} \text{ a.e. on } T. \tag{2.1}$$

We also define an auxiliary control vector $v = [v_1,v_2,\dots,v_N]^T$ and an augmented control $\tilde{u}$

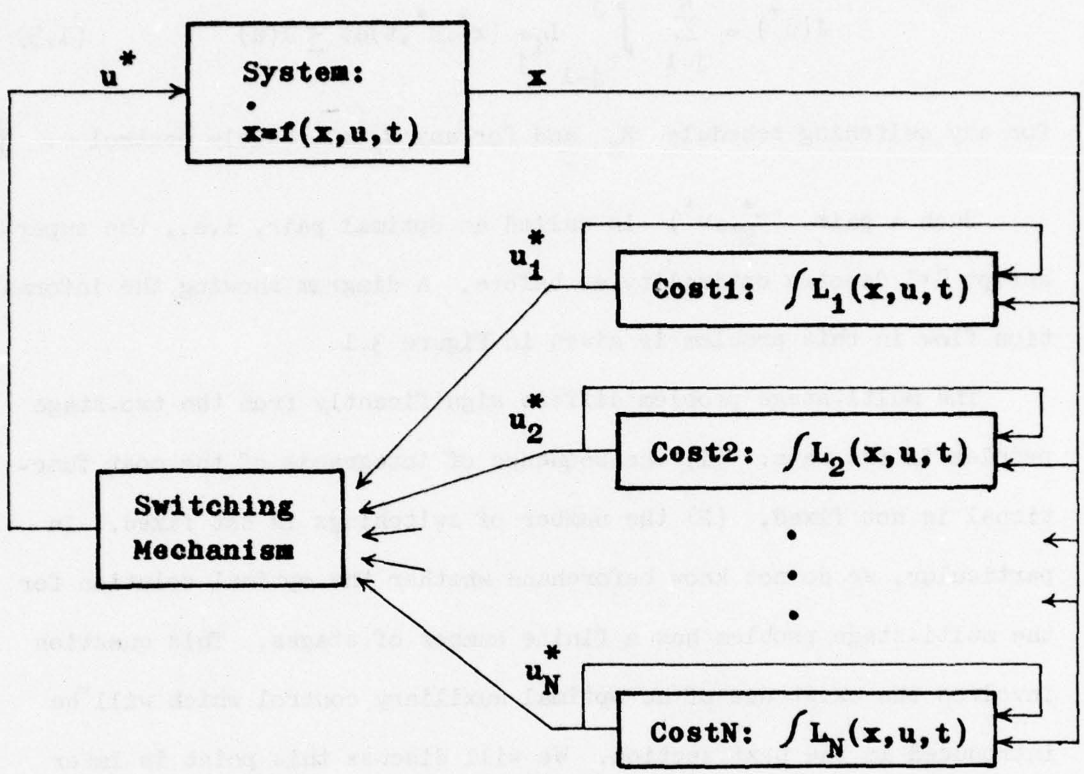

Fig. 3.1. Multi-Stage Optimal Control Problem: Schematic Diagram of Information Flow

$$\tilde{u} = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{E}^{m+N} . \qquad (2.2)$$

Let $L_0(x,\tilde{u},t)$ and $\tilde{\Omega}$ be defined as

$$L_0(x,\tilde{u},t) \triangleq \sum_{i=1}^{N} v_i L_i(x,u,t), \qquad (2.3)$$

$$\tilde{\Omega} \triangleq \bigcup_{i=1}^{N} \{(u_i,e_i) \in \mathbb{E}^{m+N} : u_i \in \Omega_i\}, \qquad (2.4)$$

where e_i denotes the unit vector in $\mathbb{E}^N$ with unit i-th component. We note here that the vector v can be considered to represent a mathematical realization of the switching mechanism shown in Fig. 3.1.

The constraint (2.1) on v_i, $i = 1,2,\ldots,N$ implies that at almost every instant of time $t \in T$, one of the $v_i(t)$'s, $i = 1,2,\ldots,N$ is equal to one and all the others are zero. Hence at almost every $t \in T$, $L_0(x,\tilde{u},t)$ is equal to one of $L_i(x,u,t)$'s, $i = 1,2,\ldots,N$, depending on which $v_i(t)$'s, $i = 1,2,\ldots,N$ is equal to one. Similarly the augmented constraint set $\tilde{\Omega}$ can be interpreted as $u(t) \in \Omega_i$ when $v_i(t) = 1$, $i = 1,2,\ldots,N$. Thus when $v_i(t) = 1$, the integrand of the cost is $L_i(x,u,t)$ and the constraint set is Ω_i, $i = 1,2,\ldots,N$. Therefore, finding an optimal auxiliary control v^* is equivalent to finding an optimal switching schedule $S^*_{K^*}$.

Now we reformulate the reduced multi-stage optimal control problem as follows.

<u>Problem (R)</u>: Given a system

$$\dot{x} = \tilde{f}(x,\tilde{u},t) \triangleq f(x,u,t) , \qquad (2.5)$$

on a control time interval (1.2) with initial and final conditions (1.3) and a cost functional

$$J(\tilde{u}) = \int_{t_0}^{t_f} L_0(x,\tilde{u},t)dt. \tag{2.6}$$

Find an admissible control $\tilde{u}^*$ which minimizes $J(\tilde{u})$, i.e., $J(\tilde{u}^*) \leq J(\tilde{u})$ for any admissible $\tilde{u}$. ‖

Note: Here we define admissibility of $\tilde{u}$ as follows.

1. $\tilde{u}$ steers x from $x(t_0) \in X_0$ to $x(t_f) \in X_f$,
2. $\tilde{u}$ is measurable on T and $\tilde{u}(t) \in \tilde{\Omega}$ a.e. on T. ‖

Problem (R) is in the standard form and the known results of control theory are directly applicable. In particular, standard necessary conditions are applicable for the characterization of $\tilde{u}^*$. We will be content with noting that the maximum principle of Appendix C is applicable to this problem. We will not give explicit details here.

On the other hand, a difficulty is encountered in ensuring the existence of $\tilde{u}^*$, due to the fact that the augmented constraint set $\tilde{\Omega}$ is never convex. Hence we may experience a situation where the optimal control is a "chattering" control. This will be discussed in detail in the following section. Furthermore, from an engineering point of view, it is not feasible to have an infinite number of switchings. That is to say, v^* should have only a finite number of discontinuities. We demand a piecewise continuous v^* rather than a measurable v^*. There have been several papers on the existence of optimal piecewise continuous controls e.g. Halkin [21], Halkin and Hendricks [22], and Grimmell [19]. However these works treated only

systems which were either linear, or linear in the state variable. The existence of optimal piecewise continuous controls for more general systems is still an open question and remains a subject for further research.

3.3 Chattering Control

For the existence of optimal measurable controls, most of the standard existence theorems require a set of augmented velocity vectors

$$V(x,t) \triangleq \left\{ \begin{bmatrix} y_0 \\ y \end{bmatrix} \in \mathbb{E}^{n+1} : y_0 = L(x,u,t),\ y = f(x,u,t),\ u \in \Omega \right\}, \tag{3.1}$$

or a set $V^+(x,t)$ defined by

$$V^+(x,t) \triangleq \left\{ \begin{bmatrix} y_0 \\ y \end{bmatrix} \in \mathbb{E}^{n+1} : y_0 \geq L(x,u,t),\ y = f(x,u,t),\ u \in \Omega \right\}, \tag{3.2}$$

to be convex for each x and t (see Appendix A). This almost exclusively requires the control constraint set Ω to be convex. In general, for problems which satisfy all except the convexity condition for the existence of u^*, an optimal control may not exist but the optimal "relaxed" control may exist for the corresponding "relaxed" problem. Warga [42] gives an extensive discussion on the relaxed problem and Berkovitz [3] has a concise presentation. Roughly speaking, the idea of the relaxed problem is that, by allowing the velocity vector $\dot{x}(t)$ to take on any value in the convex hull of $V(x,t)$, the existence of an optimal relaxed control and a corresponding relaxed trajectory is assured. In such cases, it is known that (Warga [42],

Gamkrelitze [17]) under appropriate assumptions, solutions of the relaxed problem can be uniformly approximated by solutions of the original problem. It is also known (for example, Berkovitz [3]) that such approximations can be realized by "chattering" controls u_{ch} where $u_{ch}(t)$ jumps rapidly back and forth among various points of Ω. We present a proposition which states conditions under which relaxed solutions can be uniformly approximated by ordinary solutions.

Proposition 3.1 [3]. Consider a control system defined on $T = [a,b]$, $\dot{x} = f(x,u,t)$, where f is continuous in $\mathbb{E}^n \times \mathbb{E}^m \times \mathbb{E}$. Let Ω be a control constraint set. Suppose that φ is a relaxed trajectory satisfying

$$\dot{\varphi}(t) \in \text{co } f(\varphi(t),\Omega,t), \quad \text{a.e. on } T, \tag{3.3}$$

where co E denotes the convex hull of a set E. Assume there exists an integrable function μ defined on T such that

$$\|f(x,u,t)\|_n \leq \mu(t), \tag{3.4}$$

and

$$\|f(x,u,t) - f(y,u,t)\|_n \leq \mu(t)\|x - y\|_n \tag{3.5}$$

for all $u \in \Omega$, $t \in T$. Then for any $\varepsilon > 0$, there exists an ordinary trajectory ψ satisfying

$$\dot{\psi}(t) = f(\psi(t),u(t),t), \tag{3.6}$$

corresponding to an ordinary control $u(t) \in \Omega$ a.e. on T such that

$$\|\varphi(t) - \psi(t)\|_n < \varepsilon. \quad \| \tag{3.7}$$

Returning to our problem, $\tilde{\Omega}$ given by (2.4) is compact if Ω_i, $i = 1,2,\ldots,N$ are compact but never convex. Hence, in general, we may only assume the existence of relaxed solutions for the corresponding relaxed problem and we may have to adopt a chattering control as an approximation to an optimal control.

We conclude this chapter by presenting the following example. There exists no optimal control for this example but the value of the cost can be made as close to its lower bound as desired by chattering controls. We note that if the example is formulated as a two-stage problem, then there exists an optimal control pair (u^*,t_1^*). This is also demonstrated in the example.

<u>Example</u>. Consider a multi-stage optimal control problem on $T = [0,1]$ with the system equation and terminal conditions given by

$$\dot{x} = u, \tag{3.8}$$

$$x(0) = 0, \quad x(1) \text{ free}, \tag{3.9}$$

and with a cost functional

$$J(u,v) = \int_0^1 \left[\frac{v_1}{2} \{x^2 + (u-1)^2\} + \frac{v_2}{2} \{x^2 + (u+1)^2\} \right] dt. \tag{3.10}$$

We assume that $\Omega_1 = \Omega_2 \triangleq [-2,2]$. The augmented velocity set $V(x,t)$ and the set $V^+(x,t)$ are shown in Fig. 3.2. Obviously neither set is convex. On the other hand, this example satisfies all the conditions of Proposition 3.1. Therefore the optimal control may not exist but the optimal relaxed control may exist. In that case the optimal

relaxed trajectory can be uniformly approximated by a trajectory corresponding to some chattering control. In fact, we can find a sequence of controls which do not converge to any functions in the usual sense but the sequence of the corresponding costs converges to the obvious lower limit of $J(u,v)$, namely, zero.

Note that $J(u,v)$ can be zero only when (1) $x(t) = 0$, $u(t) = 1$, $v_1(t) = 1$ and $v_2(t) = 0$, or (2) $x(t) = 0$, $u(t) = -1$, $v_1(t) = 0$ and $v_2(t) = 1$. But from (3.8), this cannot be realized. Instead, we consider the following sequence of controls (u^k,v^k):

$$u^k(t) = \left\{ \begin{array}{lll} 1 & \text{on} & [2i/2^k,\ (2i+1)/2^k] \\ -1 & \text{on} & [(2i+1)/2^k,\ 2(i+1)/2^k] \end{array} ,\ i = 0,1,\ldots,2^k-1, \right\} , \tag{3.11}$$

$$v^k(t) = \left\{ \begin{array}{lll} [1,0]^T & \text{when} & u^k(t) = 1 \\ [0,1]^T & \text{when} & u^k(t) = -1 \end{array} \right\} . \tag{3.12}$$

These controls are shown in Fig. 3.3 together with their corresponding trajectories. Obviously these controls satisfy the control constraints. From Fig. 3.3, it is clear that

$$J(u^k,v^k) = 2^k \int_0^{1/2^k} t^2/2\, dt = (3 \times 2^{2k+1})^{-1}. \tag{3.13}$$

Hence $J(u^k,v^k) \to 0$ as $k \to \infty$. But clearly as $k \to \infty$, (u^k,v^k) has no limit in the usual sense. Thus, despite the fact that there is no optimal control, $J(u,v)$ can be made as close to its lower limit zero as desired by letting the control "chatter" rapidly enough.

Now, suppose that this example was formulated as a two-stage problem rather than a multi-stage problem. In other words, we assume

that the cost is given as

$$J(u,t_1) = \int_0^{t_1} \frac{1}{2}\{x^2 + (u-1)^2\}dt + \int_{t_1}^{1} \frac{1}{2}\{x^2 + (u+1)^2\}dt \tag{3.14}$$

Then the Hamiltonians H_1 and H_2 are defined by

$$\left.\begin{aligned} H_1(x,p,u) &= -\frac{1}{2}\{x^2 + (u-1)^2\} + pu \\ H_2(x,p,u) &= -\frac{1}{2}\{x^2 + (u+1)^2\} + pu \end{aligned}\right\} . \tag{3.15}$$

The optimal controls for each stage are

$$\left.\begin{aligned} \max_u H_1(x,p,u) &\Rightarrow u^*(t) = p^*(t) + 1 \\ \max_u H_2(x,p,u) &\Rightarrow u^*(t) = p^*(t) - 1 \end{aligned}\right\} . \tag{3.16}$$

Substituting (3.16) into (3.15) leads to

$$\left.\begin{aligned} H_1^* &= -\frac{1}{2}(x^{*2} - p^{*2}) + p^* \\ H_2^* &= -\frac{1}{2}(x^{*2} - p^{*2}) - p^* \end{aligned}\right\} . \tag{3.17}$$

Now, the canonical equations for each stage are

$$\left.\begin{aligned} \frac{d}{dt}\begin{bmatrix} x^* \\ p^* \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x^* \\ p^* \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \frac{d}{dt}\begin{bmatrix} x^* \\ p^* \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x^* \\ p^* \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{aligned}\right\} . \tag{3.18}$$

This gives

$$\left.\begin{aligned} \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} &= \begin{bmatrix} (p_0^*+1)\sinh t \\ (p_0^*+1)\cosh t \end{bmatrix} \\ \begin{bmatrix} x^*(t) \\ p^*(t) \end{bmatrix} &= \begin{bmatrix} x_f^* \cosh(t-1) - \sinh(t-1) \\ x_f^* \sinh(t-1) - \cosh(t-1) + 1 \end{bmatrix} \end{aligned}\right\} , \tag{3.19}$$

where $p_0^* \triangleq p^*(0)$ and $x_f^* \triangleq x^*(1)$, and we have used $x^*(0) = 0$ and $p^*(1) = 0$. Using the transversality condition $H_1^*|(t_1^*-) = H_2^*|(t_1^*+)$, we have,

$$p^*(t_1^*) = 0. \tag{3.20}$$

From (3.19) and (3.20), t_1^*, x_f^* and p_0^* can be computed as

$$\begin{bmatrix} t_1^* \\ x_f^* \\ p_0^* \end{bmatrix} = \begin{bmatrix} 1/3 \\ -\tanh 1/3 \\ (\cosh 1/3)^{-1} - 1 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 \\ \tanh 1 \\ (\cosh 1)^{-1} - 1 \end{bmatrix}. \tag{3.21}$$

These two extremal cases are shown in Fig. 3.4. The corresponding cost values are respectively 0.02 and 0.12. Thus, an optimal control pair (u^*, t_1^*) is given by

$$t_1^* = 1/3, \tag{3.22}$$

$$u^*(t) = \begin{Bmatrix} (\cosh t)/(\cosh 1/3), & 0 \leq t < 1/3 \\ -(\cosh(t-2/3))/(\cosh 1/3), & 1/3 \leq t \leq 1 \end{Bmatrix}. \tag{3.23}$$

We note that $u^*(t)$ satisfies the control constraint. The corresponding trajectories $x^*(t)$ and $p^*(t)$ are

$$x^*(t) = \begin{Bmatrix} (\sinh t)/(\cosh 1/3), & 0 \leq t \leq 1/3 \\ -(\sinh(t-3/2))/(\cosh 1/3), & 1/3 \leq t \leq 1 \end{Bmatrix}, \tag{3.24}$$

$$p^*(t) = \begin{Bmatrix} u^*(t) - 1, & 0 \leq t < 1/3 \\ u^*(t) + 1, & 1/3 \leq t \leq 1 \end{Bmatrix}. \tag{3.25}$$

This concludes the example.

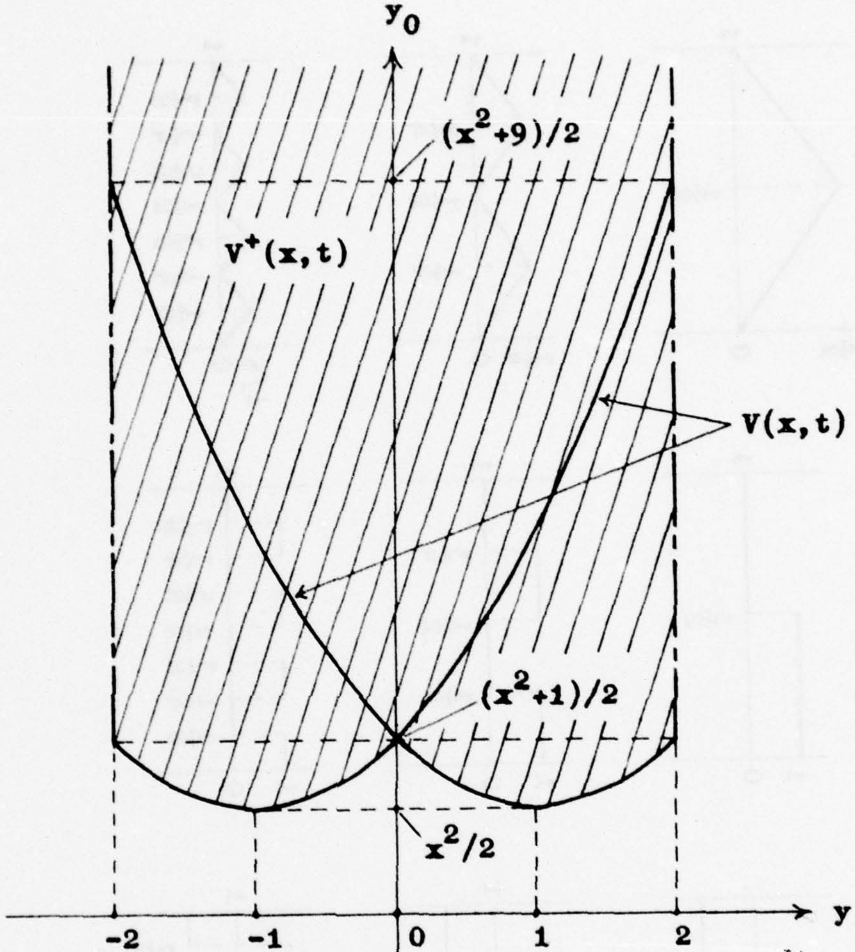

Fig. 3.2. Sets $V(x,t)$ and $V^+(x,t)$

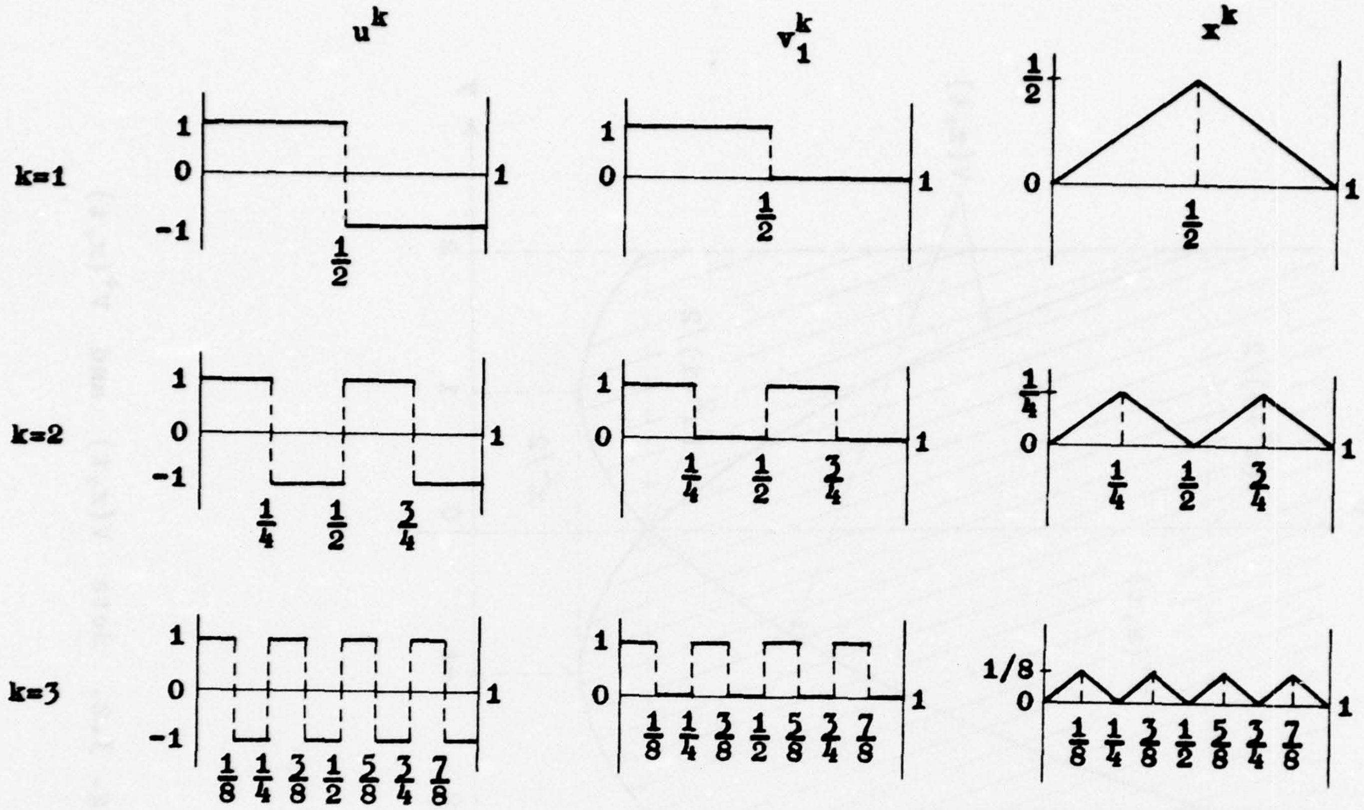

Fig. 3.3. Control Sequence and Corresponding Trajectories

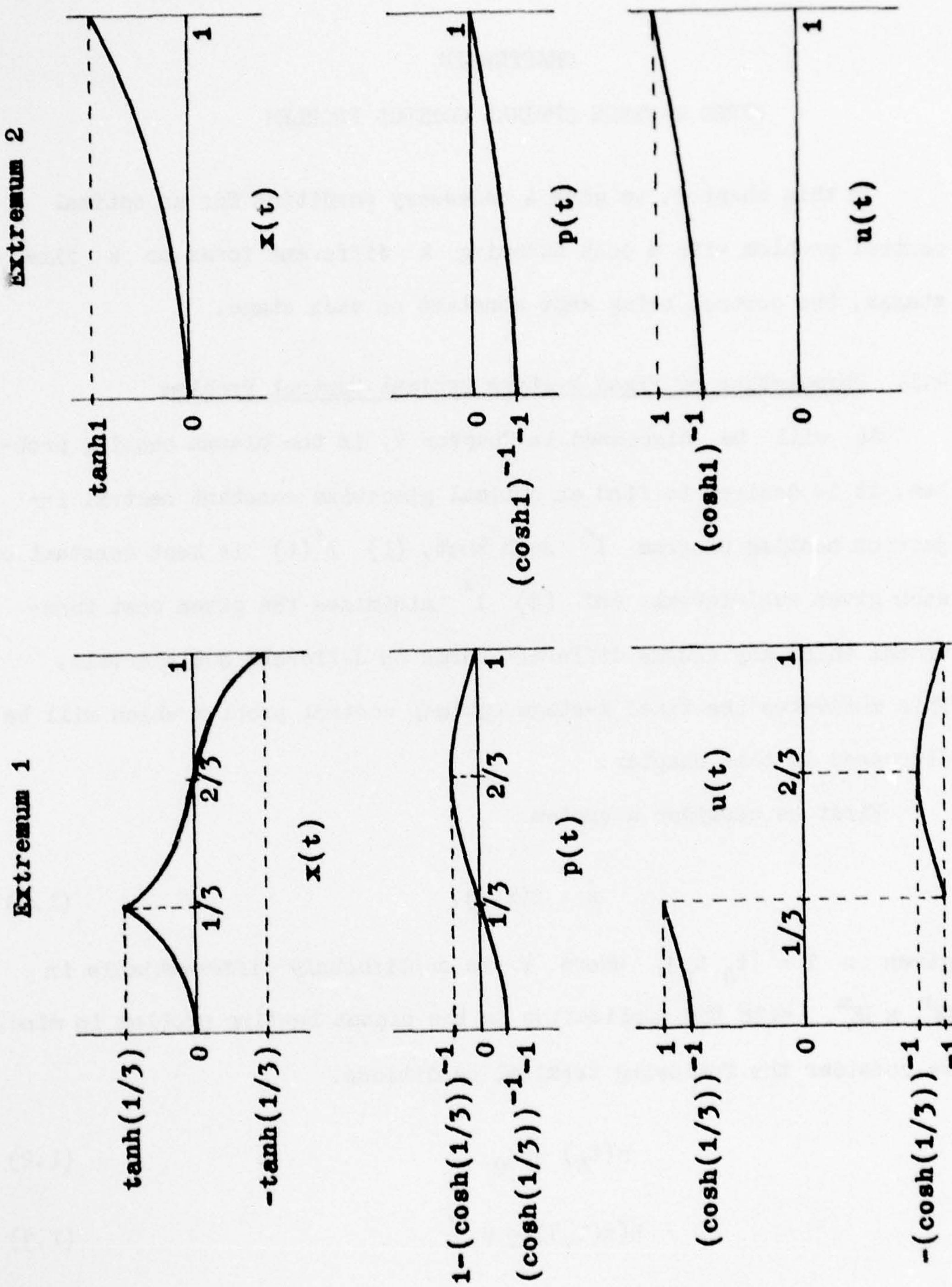

Fig. 3.4. Extremal Controls and Corresponding Trajectories and Adjoints

CHAPTER IV

FIXED k-STAGE OPTIMAL CONTROL PROBLEM

In this chapter, we give a necessary condition for an optimal control problem with a cost assuming k different forms on k fixed stages, the control being kept constant on each stage.

4.1. Formulation of Fixed k-stage Optimal Control Problem

As will be discussed in Chapter V, in the plasma heating problem, it is desired to find an optimal piecewise constant neutral injection heating program I^* such that, (1) $I^*(t)$ is kept constant on each given subinterval, and (2) I^* minimizes the given cost functional which may assume different forms on different subintervals. This motivates the fixed k-stage optimal control problem which will be discussed in this chapter.

First we consider a system

$$\dot{x} = f(x,u), \tag{1.1}$$

given on $T = [t_0,t_f]$, where f is continuously differentiable in $\mathbb{E}^n \times \mathbb{E}^m$. With the application to the plasma heating problem in mind, we consider the following terminal conditions.

$$x(t_0) = x_0, \tag{1.2}$$

$$h(x(t_f)) \leq 0, \tag{1.3}$$

where x_0 is given and h is a continuously differentiable function of $x(t_f)$. Let $\{\Omega_i, L_i(\cdot,\cdot)\}$, $i = 1,2,\ldots,k$ be k given pairs of

control constraint sets and integrands of the cost. We assume that $\Omega_i \subseteq \mathbb{E}^m$ are nonempty convex compact sets and the L_i are continuously differentiable in $\mathbb{E}^n \times \mathbb{E}^m$. Then the fixed k-stage cost is defined as

$$J(u) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} L_i(x(t),u(t))dt \tag{1.4}$$

where t_i, $i = 1,2,\ldots,k-1$ are given switching times and $t_k = t_f$.

<u>Definition 1.1</u>. A control u is said to be <u>admissible</u> if

1. there exists a corresponding solution of (1.1) which satisfies the terminal conditions (1.2) and (1.3),
2. $u(t)$ is constant and $u(t) = u_i \in \Omega_i$ on each $[t_{i-1},t_i)$, $i = 1,2,\ldots,k-1$, and $u(t) = u_k \in \Omega_k$ on $[t_{k-1},t_f]$. ||

Now the fixed k-stage optimal control problem is stated as follows:

<u>Problem (F)</u>: Given system (1.1), terminal conditions (1.2) and (1.3) and k pairs of constraint sets and integrands of the cost functional $\{\Omega_i,L_i\}$, $i = 1,2,\ldots,k$, find an admissible piecewise constant control u^* which minimizes the cost (1.4), i.e., $J(u^*) \leq J(u)$ for any admissible u. ||

Without loss of generality, we consider the case $u(t) \in \mathbb{E}^1$ only. Also we assume that the constraint sets are compact intervals, i.e.,

$$\Omega_i = [v_i,w_i], \qquad i = 1,2,\ldots,k, \tag{1.5}$$

where $-\infty < v_i \leq w_i < \infty$, $i = 1,2,\ldots,k$. The results obtained in the following sections can be easily extended to the case with m-dimensional controls.

4.2. Reformulation as Parameter Optimization Problem

For the fixed k-stage optimal control problem, it is required that the control function is constant on each fixed subinterval. Since there are k fixed subintervals, the control function can be written as

$$u(t) = \sum_{i=1}^{k-1} u_i \chi_{[t_{i-1},t_i)} + u_k \chi_{[t_{k-1},t_k]}, \tag{2.1}$$

where χ_E is the characteristic function of a set E. By considering (2.1) as a map $\mathfrak{F}$ from $\mathbb{E}^k$ to the class of control functions under consideration, we can identify the control functions with vectors in $\mathbb{E}^k$. Therefore we can formulate the fixed k-stage optimal control problem as a parameter optimization problem in $\mathbb{E}^k$.

Let $\underline{u}$ denote the parameter vector $[u_1,u_2,\ldots,u_k]^T \in \mathbb{E}^k$. Then the constraint on the control function

$$u(t) = \left\{ \begin{array}{ll} u_i \in \Omega_i = [v_i,w_i] & \text{on} \quad [t_{i-1},t_i),\ i=1,2,\ldots,k-1, \\ u_k \in \Omega_k = [v_k,w_k] & \text{on} \quad [t_{k-1},t_f], \end{array} \right\} \tag{2.2}$$

is equivalent to

$$g_i(\underline{u}) \triangleq (u_i - v_i)(u_i - w_i) \leq 0, \ i=1,2,\ldots,k. \tag{2.3}$$

Next, the final condition (1.3) can be transcribed as a constraint on the parameter vector $\underline{u}$ by considering a map from $\underline{u}$ to $x(t_f)$ such that

$$x_k(\underline{u}) \triangleq x(t_f) \qquad (2.4)$$

where $x(t_f)$ is the final point of the trajectory x corresponding to the control function $\mathfrak{J}(\underline{u})$. Substituting (2.4) into (1.3) yields

$$h(x_k(\underline{u})) \leq 0. \qquad (2.5)$$

Now the fixed k-stage optimal control problem is reformulated as a parameter optimization problem.

Problem (P): Minimize the cost $J(\underline{u})$ given by

$$J(\underline{u}) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} L_i(x(t),u_i)dt, \qquad (2.6)$$

with respect to $\underline{u} = [u_1,u_2,\ldots,u_k]$ subject to (2.3) and (2.5), where $x(t)$ is a solution trajectory of (1.1) with (1.2) and a control function $\mathfrak{J}(\underline{u})$. ||

Thus we have a standard nonlinear programming problem for which the standard results of nonlinear programming are readily applicable.

4.3. Necessary Condition for Optimal Parameter Vector

In this section, we present a set of necessary conditions for an optimal parameter vector by applying the Kuhn-Tucker (K-T) condition to the reformulated parameter optimization problem. The K-T condition in general form is given in Appendix D.

First we define the Lagrangians $\tilde{J}(\underline{u})$ and $M(\underline{u})$ as

$$\tilde{J}(\underline{u}) \triangleq \lambda_0 J(\underline{u}) + \mu h(x_k(\underline{u})), \qquad (3.1)$$

$$M(\underline{u}) \triangleq \tilde{J}(\underline{u}) + \sum_{i=1}^{k} \lambda_i g_i(\underline{u}), \qquad (3.2)$$

where λ_i, $i = 0,1,\ldots,k$ and μ are scalar multipliers. Let $\underline{u}^*$ be an optimal parameter vector. Then the K-T condition states that at $\underline{u} = \underline{u}^*$

$$\nabla_u M(\underline{u}^*) = \nabla_u \tilde{J}(\underline{u}^*) + \sum_{i=1}^{k} \lambda_i \nabla_u g_i(\underline{u}^*) = 0. \tag{3.3}$$

As we have discussed in Chapter I, no assumption will be introduced in deriving the gradients in (3.3).

The following theorem gives necessary conditions in integral form. We note that the gradient $\nabla_{\underline{u}} M$ can be written in a compact form by introducing the adjoint vector $p(t)$ and the Hamiltonians $H_i(x(t),p(t),u_i,\lambda_0)$, $i = 1,2,\ldots,k$.

Theorem 3.1. Suppose a vector $\underline{u}^* = [u_1^*,u_2^*,\ldots,u_k^*]^T$ is optimal, then it is necessary that

$$\left\{\begin{array}{l} \dot{p}^*(t) = -\{\partial H_i/\partial x\, (x^*(t),p^*(t),u_i^*,\lambda_0^*)\}^T \\ \qquad \text{on } (t_{i-1},t_i),\ i = 1,2,\ldots,k, \\ p^*(t_i-) = p^*(t_i+),\quad i = 1,2,\ldots,k-1, \\ p^*(t_f) = -\mu^*\{\partial h(x_k(\underline{u}^*))/\partial x_k\}^T, \end{array}\right\} \tag{3.4}$$

where $x^*(t)$ satisfies

$$\left\{\begin{array}{l} \dot{x}^*(t) = \{\partial H_i/\partial p(x^*(t),p^*(t),u_i^*,\lambda_0^*)\}^T \\ \qquad \text{on } (t_{i-1},t_i),\ i = 1,2,\ldots,k, \\ x^*(t_i-) = x^*(t_i+),\quad i = 1,2,\ldots,k-1, \\ x^*(t_0) = x_0,\quad h(x^*(t_f)) \le 0, \end{array}\right\} \tag{3.5}$$

and

$$H_i(x^*(t),p^*(t),u_i^*,\lambda_0^*) \overset{\Delta}{=} \lambda_0^* L_i(x^*(t),u_i^*) + p^*(t)^T f(x^*(t),u_i^*), \tag{3.6}$$

$$i = 1,2,\ldots,k;$$

2. there exist scalar nonpositive multipliers $\lambda_i^* \le 0$, $i = 1,2,\ldots,k$ such that

$$\lambda_i^* g_i(\underline{u}^*) = 0, \quad i = 1,2,\ldots,k; \tag{3.7}$$

3.

$$\nabla_{\underline{u}} M(\underline{u}^*) = \begin{bmatrix} \int_{t_0}^{t_1} \partial H_1/\partial u_1(x^*(t),p^*(t),u_1^*,\lambda_0^*)dt + 2\lambda_1^*(u_1^* - \bar{u}_1) \\ \int_{t_1}^{t_2} \partial H_2/\partial u_2(x^*(t),p^*(t),u_2^*,\lambda_0^*)dt + 2\lambda_2^*(u_2^* - \bar{u}_2) \\ \vdots \\ \int_{t_{k-1}}^{t_k} \partial H_k/\partial u_k(x^*(t),p^*(t),u_k^*,\lambda_0^*)dt + 2\lambda_k^*(u_k^* - \bar{u}_k) \end{bmatrix} = 0 \tag{3.8}$$

where $\bar{u}_i = (v_i + w_i)/2, \quad i = 1,2,\ldots,k.$ ||

For the proof of this theorem, we refer to Appendix F.

Remarks:

(R.1) μ^* in (3.4) is a nonpositive multiplier corresponding to the condition $h(x_k(\underline{u}^*)) \le 0$. Moreover, $\mu^* < 0$ if $h(x_k(\underline{u}^*)) = 0$ and $\mu^* = 0$ if $h(x_k(\underline{u}^*)) < 0$.

(R.2) The multiplier λ_0^* in (3.6) is nonpositive. We can assume $\lambda_0^* < 0$ if the first-order constraint qualification is satisfied (various types of constraint qualifications are discussed in Mangasarian [31]). But when the constraint is of the implicit type such as (2.5), it is generally impossible to check the qualification *a priori*. ||

The following theorem gives a more explicit characterization of $\underline{u}^*$. It is said that the constraint g_i is active if $g_i(\underline{u}) \leq 0$ is satisfied by equality.

<u>Theorem 3.2</u>. Let $\underline{u}^* = [u_1^*, u_2^*, \ldots, u_k^*]^T$ be optimal. Suppose $\int_{t_{i-1}}^{t_i} (\partial H_i^*/\partial u_i)(t)dt \neq 0$, then g_i is active and

$$u_i^* = \begin{cases} v_i, & \text{if } \int_{t_{i-1}}^{t_i} (\partial H_i^*/\partial u_i)(t)dt < 0 \\ w_i, & \text{if } \int_{t_{i-1}}^{t_i} (\partial H_i^*/\partial u_i)(t)dt > 0 \end{cases}, \tag{3.9}$$

where $(\partial H_i^*/\partial u_i)(t) \stackrel{\Delta}{=} (\partial H_i/\partial u_i)(x^*(t), p^*(t), u_i^*, \lambda_0^*)$. ||

<u>Remark</u>: When $\lambda_i^* = 0$, it is obvious that

$$\int_{t_{i-1}}^{t_i} \partial H_i^*/\partial u_i(t)dt = 0. \tag{3.10}$$

For this case u_i^* may take any value in $[v_i, w_i]$ and the equality (3.10) must be solved to find u_i^*. ||

<u>Proof of Theorem 3.2</u>. Suppose

$$\int_{t_{i-1}}^{t_i} (\partial H_i^*/\partial u_i)(t)dt > 0. \tag{3.11}$$

Then in order for (3.8) to be satisfied, we must have

$$\lambda_i^*(u_i^* - \bar{u}_i) < 0. \tag{3.12}$$

This implies $\lambda_i^* \neq 0$ and, from the nonpositiveness of λ_i^*, $\lambda_i^* < 0$. Now, for $\lambda_i^* < 0$, the condition (3.7) states that g_i is active, i.e. $g_i(\underline{u}^*) = 0$. Hence either $u_i^* = v_i$ or $u_i^* = w_i$. But again for $\lambda_i^* < 0$,

(3.12) can be satisfied only if $u_i^* - \bar{u}_i > 0$. Therefore $u_i^* = w_i$. The other case can be shown similarly. ||

In general, some iterative procedure must be employed for the computation of $\underline{u}^*$ since the conditions in the above theorems involve $\underline{u}^*$. For the computational aspects of the nonlinear programming, we refer to standard texts (for example, Canon, Cullum and Polak [6]).

CHAPTER V

PLASMA HEATING BY NEUTRAL INJECTION

A problem of minimum input energy plasma heating by neutral injection is presented in this chapter. The problem is formulated as a two-stage optimal control problem and is solved utilizing the results of Chapter II. An optimal piecewise constant neutral injection program is then derived using the techniques discussed in Chapter IV.

5.1. Minimum Input Energy Plasma Heating Problem

In a recent report [9], the problem of toroidal plasma heating by neutral injection was studied using a spatially-averaged two-temperature model of the plasma. It was shown that the problem of minimizing the total input energy while achieving a desired ion temperature gives rise to a two-stage optimal control problem. We will give an explicit formulation of this problem in this section.

When the electron and ion temperatures T_e and T_i are of the order of KeV, we have the following equations for the average energy transport associated with the electrons and ions.

$$\frac{3}{2}\frac{dT_e}{dt} = S_J - S_{De} - S_{ei} - S_R + S_N^e, \tag{1.1a}$$

$$\frac{3}{2}\frac{dT_i}{dt} = S_{ei} - S_{Di} - S_{ex} + S_N^i , \tag{1.1b}$$

where S_J, S_{ei}, S_{Dj}, S_R, S_{ex} and S_N^j are respectively the energy transport terms corresponding to Joule heating, electron-ion energy exchange, diffusion loss of species j (j = e and i for electrons

and ions), line radiation loss, excitation and power-transfer from neutral beam to species j. In explicit form, (1.1) can be expressed as

$$\frac{dT_e}{dt} = A_1 I_p^2 T_e^{-3/2} - A_2 I_p^{-\beta} T_e^{\alpha} - A_3 (T_e - T_i) T_e^{-3/2} - A_4 + A_5 X_1, \qquad (1.2a)$$

$$\frac{dT_i}{dt} = A_3 (T_e - T_i) T_e^{-3/2} - S_{Di}(T_i, I_p) - B_4 T_i + B_5 X_2, \qquad (1.2b)$$

$$\frac{dX_1}{dt} = T_e^{-3/2}(EI) - (C_1 T_e^{-3/2} + C_2 E^{-3/2}) X_1, \qquad (1.2c)$$

$$\frac{dX_2}{dt} = E^{-3/2}(EI) - (C_1 T_e^{-3/2} + C_2 E^{-3/2}) X_2, \qquad (1.2d)$$

where I_p is the toroidal plasma current, E and I are the average particle kinetic-energy and current of the injected neutral beam, and X_1 and X_2 are normalized injected power transferred to electrons and ions respectively. The average particle kinetic-energy E is assumed to be fixed. The exponents α and β depend on the electron diffusion regime. Here we set $\alpha = 1/2$ and $\beta = 0$ which correspond to the collisional regime. The ion diffusion in the three regimes is approximated by the function S_{Di} defined by

$$S_{Di}(T_i, I_p) = \mathrm{Int}\{B_1 T_i^{1/2} I_p^{-2}, B_2 T_i^{5/2} I_p^{-1}, B_3 T_i^{1/2} I_p^{-2}\}, \qquad (1.3)$$

where $\mathrm{Int}\{a_1, a_2, a_3\}$ denotes the intermediate value of the three quantities a_i, $i = 1,2,3$. The coefficients A_j, B_k and C_ℓ are given in Appendix F. For a detailed discussion of this two-temperature model, we refer to [9].

As was discussed in Chapter I, we consider a heating program beginning with an initial Joule heating period followed by combined

neutral injection and Joule heating.

Let the heating interval $[0,t_f]$ be fixed. For a neutral injection current $I(t)$ defined on $[t_1,t_f]$ with $t_1 \in [0,t_f]$, the total input energy into the plasma is given by

$$J(I,t_1) = \int_0^{t_f} A_1 I_p^2 T_e^{-3/2}(t)dt + \int_{t_1}^{t_f} REI(t)dt, \tag{1.4}$$

where the integrands of the first and second integrals correspond to Joule heating and neutral injection heating respectively, and the constant R is given in Appendix F.

<u>Minimum Energy Plasma Heating Problem</u>: Given the two-temperature model of the plasma described by (1.2); a finite heating time interval $T \triangleq [0,t_f]$; initial conditions

$$(T_e(0),T_i(0),X_1(0),X_2(0)) = (T_{e0},T_{i0},0,0); \tag{1.5}$$

a desired ion temperature $T_{id} > T_{i0}$; and a constraint on the injection current $I(t)$ of the form

$$\begin{cases} I(t) = 0 & \text{on } [0,t_1), \\ I(t) \in \Omega_2 \triangleq [I_{min},I_{max}] \text{ a.e.} & \text{on } [t_1,t_f], \end{cases} \tag{1.6}$$

where $T_{e0} > 0$, $T_{i0} > 0$, $I_{min} \geq 0$ and $I_{max} > I_{min}$ are specified finite constants, find a neutral injection program (I^*,t_1^*) such that $T_i(t_f) \geq T_{id}$ and the total input energy $J(I,t_1)$ given by (1.4) is minimized. ||

Note that in this problem formulation, E is held constant. In general, both E and I can be considered as controls.

We rewrite the total cost J in the following form.

$$J(I,t_1) = \int_0^{t_1} A_1 I_p^2 T_e^{-3/2}(t)dt + \int_{t_1}^{t_f} \{A_1 I_p^2 T_e^{-3/2}(t) + REI(t)\}dt. \tag{1.7}$$

Thus, the minimum input energy plasma heating problem is formulated as a two-stage optimal control problem for which the results of Chapter II are readily applicable.

5.2. Existence of an Optimal Heating Program

We shall show that the plasma heating problem satisfies all the conditions for the existence of an optimal control pair (u^*,t_1^*) given in Theorems II.2.1 and II.2.2 .

Let x denote the system state vector $[T_e,T_i,X_1,X_2]^T \in \mathbb{E}^4$. Let the right hand side of the system equation (1.2a)-(1.2b) be denoted by $f(x,I) = [f_e,f_i,f_1,f_2]^T$.

The initial set X_0 and the constraint sets Ω_1 and Ω_2 are given by

$$X_0 = [T_{e0},T_{i0},0,0]^T, \quad \Omega_1 = \{0\}, \quad \Omega_2 = [I_{min},I_{max}]. \tag{2.1}$$

Hence the compactness condition (1) in Theorem II.2.1 is satisfied. The convexity condition (5) in the same theorem is also satisfied since the augmented velocity vectors $(A_1 I_p^2 T_e^{-3/2}, f(x,I))$ and $(A_1 I_p^2 T_e^{-3/2} + REI, f(x,I))$ are affine in the control I, Ω_1 is a point and Ω_2 is convex. Note that Theorem II.2.1 is still valid when the set of augmented velocity vectors is reduced to a point.

We note that the two-temperature model (1.2) of the plasma is defined only in a cone $\mathbf{C} \in \mathbb{E}^4$ given by

$$C = \{[T_e, T_i, X_1, X_2]^T : 0 < T_e, 0 \leq T_i, 0 \leq X_1, 0 \leq X_2\}. \quad (2.2)$$

Let $\mathcal{B}$ be a nonempty compact subset of $\mathbf{C}$. Then $f(\mathbf{x}, I)$ and the integrands of the cost functional (1.4) are uniformly continuous in $\mathcal{B} \times \{\Omega_1 \cup \Omega_2\} \times T$. This shows that the continuity and boundedness condition (3) of Theorem II.2.1 is satisfied on $\mathcal{B} \times \{\Omega_1 \cup \Omega_2\} \times T$ for any compact set $\mathcal{B} \subset C$.

Now the measurability of the control $I(\cdot)$ and the boundedness of $f(\mathbf{x}, I)$ implies that $F(\mathbf{x}, t) \triangleq f(\mathbf{x}, I(t))$ satisfies the Caratheodory hypothesis [7]. Furthermore, $F(x,t)$ satisfies the Lipschitz condition

$$\|F(x,t) - F(y,t)\|_n \leq M_d \|x - y\|_n, \quad (2.3)$$

for any x and y in any compact set $\mathcal{B} \subset C$ and any fixed $t \in T$, because the Jacobian matrix $(\partial F(x,t)/\partial x)$ exists for almost all $\mathbf{x} \in C$ for fixed $t \in T$, and each element $(\partial F/\partial \mathbf{x})_{ij}$, $i,j = 1,2,3,4$ is uniformly bounded in any $\mathcal{B} \subset C$, i.e., for any $x \in \mathcal{B}$ and $t \in T$,

$$|(\partial F(\mathbf{x},t)/\partial \mathbf{x})_{ij}| \leq M_{ij} < \infty, \quad i,j = 1,2,3,4, \quad (2.4)$$

whenever $(\partial F(\mathbf{x},t)/\partial \mathbf{x})_{ij}$ exists. Thus we can apply Theorem 2.2 in Chapter 2 of [7] to establish the existence of a unique solution of (1.2) passing through any point of $\mathcal{B}$ with any control satisfying the control constraint. Consequently, condition (2) of Theorem II.2.1 of the existence of a unique solution is satisfied.

Next we shall prove the following theorem, which shows that condition (4) of Theorem II.2.1 is satisfied.

Theorem 2.1. There exists a compact set $\mathcal{S} \subset \mathcal{C}$ such that (1) the given initial point $x_0 = [T_{e0}, T_{i0}, 0, 0]^T$ is in $\mathcal{S}$, and (2) for any control I which satisfies the control constraint (1.6), the corresponding trajectory does not escape from $\mathcal{S}$. ||

For the proof of this theorem, we need the following result.

Lemma 2.1. Consider a function g defined on $(0,\infty)$ such that

$$g(r) = A_1 I_p^2 r^{-3/2} - A_2 r^{1/2} - A_3 r^{-1/2} - A_4 - \varepsilon, \tag{2.5}$$

where $\varepsilon > 0$ is a constant. Assume that $A_3^2 - 12A_1A_2I_p^2 < 0$. Then $g(r)$ is strictly monotone decreasing and $g(r) = 0$ has a unique solution in $(0,\infty)$. ||

Proof: Since $g(r)$ is continuous in $(0,\infty)$ and

$$\lim_{r\to 0+} g(r) = \infty, \qquad \lim_{r\to\infty} g(r) = -\infty, \tag{2.6}$$

there exists at least one solution for $g(r) = 0$. On the other hand,

$$\partial g/\partial r = -r^{-5/2}(A_2 r^2 - A_3 r + 3A_1 I_p^2)/2. \tag{2.7}$$

Since the discriminant D of the parenthesized quadratic form is negative by assumption, we have

$$(\partial g/\partial r) < 0, \qquad \forall\, r > 0. \tag{2.8}$$

Thus $g(r)$ is strictly monotone decreasing and there exists a unique solution for $g(r) = 0$ in $(0,\infty)$. ||

Note that for the actual parameter values (see Appendix G), $D \approx -9$. Hence the condition in Lemma 2.1 is satisfied.

Now we proceed to prove Theorem 2.1.

Proof of Theorem 2.1: Let T_{ec} be the unique solution of $g(r) = 0$. Let $T_{emin} = \min\{T_{ec}, T_{e0}\}$. Hereafter, we assume that $T_e \geq T_{emin}$. For any $I \in [I_{min}, I_{max}]$ or $I = 0$, we have

$$f_1|_{X_1=0} = T_e^{-3/2} EI \geq 0. \tag{2.9}$$

Let X_{1max} be given by

$$X_{1max} \triangleq 1 + (T_{emin}^{-3/2} EI_{max})/(C_2 E^{-3/2}). \tag{2.10}$$

Then for any I satisfying the constraint,

$$\begin{aligned} f_1|_{X_1=X_{1max}} &= T_e^{-3/2} EI - (C_1 T_e^{-3/2} + C_2 E^{-3/2}) X_{1max} \\ &\leq T_{emin}^{-3/2} EI_{max} - C_2 E^{-3/2}\{1 + (T_{emin}^{-3/2} EI_{max})/(C_2 E^{-3/2})\} \\ &= -C_2 E^{-3/2} < 0. \end{aligned} \tag{2.11}$$

Next we consider f_2. For any $I \in [I_{min}, I_{max}]$ or $I = 0$,

$$f_2|_{X_2=0} = T^{-3/2} EI \geq 0. \tag{2.12}$$

Define X_{2max} as

$$X_{2max} \triangleq 1 + EI_{max}/C_2. \tag{2.13}$$

Then again for any $I \in [I_{min}, I_{max}]$ or $I = 0$,

$$f_2\big|_{X_2=X_{2max}} = E^{-3/2}(EI) - (C_1T_e^{-3/2} + C_2E^{-3/2})X_{2max}$$

$$\leq E^{-3/2}(EI_{max}) - C_2E^{-3/2}(1 + EI_{max}/C_2)$$

$$= -C_2E^{-3/2} < 0. \tag{2.14}$$

Furthermore, for $X_2 \in [0, X_{2max}]$,

$$f_i\big|_{T_i=0} = A_3T_e^{-1/2} + B_5X_2 \geq 0. \tag{2.15}$$

We also have for $X_2 \in [0, X_{2max}]$

$$f_i\big|_{T_i=T_{imax}} = A_3T_e^{-1/2} - A_3T_e^{-3/2}T_{imax} - S_{Di}(T_{imax}, I_p) + B_4T_{imax} + B_5X_2$$

$$< A_3T_{emin}^{-3/2} - B_4T_{imax} + B_5X_{2max} = 0, \tag{2.16}$$

where T_{imax} is given by

$$T_{imax} \stackrel{\Delta}{=} (A_3T_{emin}^{-1/2} + B_5X_{2max})/B_4. \tag{2.17}$$

Now we compute $f_e\big|_{T_e=T_{emin}}$. From Lemma 2.1,

$$f_e\big|_{T_e=T_{emin}} = (A_1I_p^2 + A_3T_i)T_{emin}^{-3/2} - A_2T_{emin}^{1/2} - A_3T_{emin}^{-1/2} - A_4 + A_5X_1$$

$$\geq A_1I_p^2T_{emin}^{-3/2} - A_2T_{emin}^{1/2} - A_3T_{emin}^{-1/2} - A_4 \geq \varepsilon > 0, \tag{2.18}$$

for any $T_i \in [0, T_{imax}]$ and $X_1 \in [0, X_{1max}]$ since $T_{emin} \leq T_{ec}$ and $g(T_{emin}) \geq g(T_{ec})$. Finally, let $T_{emax} > 0$ be a solution of

$$(A_1I_p^2 + A_3T_{imax})T_{emax}^{-3/2} - A_2T_{emax}^{1/2} + A_5X_{1max} = 0. \tag{2.19}$$

Then for any $T_i \in [0,T_{imax}]$ and $X_1 \in [0,X_{1max}]$,

$$
\begin{aligned}
f_e\big|_{T_e=T_{emax}} &= (A_1I_p^2+A_3T_i)T_{emax}^{-3/2}-A_2T_{emax}^{1/2}-A_3T_{emax}^{-1/2}-A_4+A_5X_1 \\
&< (A_1I_p^2+A_3T_{imax})T_{emax}^{-3/2}-A_2T_{emax}^{1/2}-A_4+A_5X_{1max} \\
&= -A_4 < 0. \qquad (2.20)
\end{aligned}
$$

Thus we have established that there exists no trajectory which escapes from the region

$$
\begin{aligned}
\mathcal{S} \triangleq \{[T_e,T_i,X_1,X_2]^T : T_{emin} \le T_e \le T_{emax},\ 0 \le T_i \le T_{imax}, \\
0 \le X_1 \le X_{1max}, 0 \le X_2 \le X_{2max}\}. \qquad (2.21)
\end{aligned}
$$

The given initial point $x_0 = [T_{e0},T_{i0},0,0]^T$ is in $\mathcal{S}$ since $T_{emin} \le T_{e0}$ and $0 \le T_{i0}$. Consequently, any trajectory initiating from x_0 at time $t = 0$ remains in $\mathcal{S}$. $\|$

Theorem 2.1 establishes the last remaining condition (4) of Theorem II.2.1. Thus, we only have to show condition (6) in Theorem II.2.2 in order to conclude the existence of an optimal heating program.

Now we make the following two assumptions:

1. $T_i(t_f;I_{max}) > T_{id}$, where $T_i(t_f;I_{max})$ is the final ion temperature with $I(t) = I_{max}$ for all $t \in [0,t_f]$;
2. $T_{id} > \overline{T}_i$, where $\overline{T}_i$ is the equilibrium ion temperature corresponding to Joule heating only.

The first assumption is essential in order for our problem to be meaningful. Otherwise the desired temperature T_{id} can never be

achieved. The second assumption refers to the actual requirement that T_{id} is near the ignition temperature. It is implied in this assumption that neutral injection must be introduced over some positive time duration so that the ion temperature can reach T_{id}. In [9] it was shown that the time-optimal heating program which steers the system from x_0 to the target set $\{x = [T_e, T_i, X_1, X_2]^T : T_i \geq T_{id}\}$ in the smallest amount of time is given by $I(t) = I_{max}$ for all t. This implies that there exists a critical switching time t_{1c} for which there exists only one trajectory with $I(t) = I_{max}$ on $[t_{1c}, t_f]$ such that $T_i(t_f) = T_{id}$, and there is no trajectory with $T_i(t_f) \geq T_{id}$ for any switching time $t_1 > t_{1c}$. In other words, the set Δ_{t_1} of t_1-admissible controls is nonempty for $t_1 \in [0, t_{1c}]$. This shows that condition (6) in Theorem II.2.2 is satisfied.

Thus the existence of an optimal heating program (I^*, t_1^*) is established.

5.3. Characterization of Optimal Heating Program

In this section, we apply Theorem 4.1 of Chapter II to the plasma heating problem and derive necessary conditions for an optimal heating program.

First we form the Hamiltonians H_i, $i = 1,2$.

$$H_1(x,p,I) = -A_1 I_p^2 T_e^{-3/2} + p_1 f_e + p_2 f_i + p_3 f_1 + p_4 f_2$$

$$= (p_3 T_e^{-3/2} E + p_4 E^{-1/2}) I + M, \qquad (3.1)$$

$$H_2(x,p,I) = -(A_1 I_p^2 T_e^{-3/2} + REI) + p_1 f_e + p_2 f_i + p_3 f_1 + p_4 f_2 = S_W I + M, \tag{3.2}$$

where $p = [p_1, p_2, p_3, p_4]^T$ is the adjoint vector and $S_W(t)$ and M are defined by

$$S_W(t) \triangleq Ep_3(t) T_e^{-3/2}(t) + E^{-1/2} p_4(t) - RE, \tag{3.3}$$

$$M \triangleq p_1 f_e + p_2 f_i + (p_3 X_1 + p_4 X_2)(C_1 T_e^{-3/2} + C_2 E^{-3/2}) - A_1 I_p^2 T_e^{-3/2}. \tag{3.4}$$

Since the constraint sets Ω_1 and Ω_2 are given by

$$\Omega_1 = \{0\}, \quad \Omega_2 = [I_{min}, I_{max}], \tag{3.5}$$

the maximization of H_1 and H_2 can be performed as

$$\max_{I \in \Omega_1} H_1(x^*, p^*, I) = H_1(x^*, p^*, 0) = M^*(t), \tag{3.6}$$

$$\max_{I \in \Omega_2} H_2(x^*, p^*, I) = \begin{cases} S_W^*(t) I_{min} + M^*(t), & \text{if } S_W^*(t) < 0 \\ S_W^*(t) I_{max} + M^*(t), & \text{if } S_W^*(t) > 0 \end{cases}. \tag{3.7}$$

The transversality condition $H_1^*|_{t_1^*} = H_2^*|_{t_1^*}$ gives

$$M^*(t_1^*-) = \begin{cases} S_W^*(t_1^*+) I_{min} + M^*(t_1^*+), & \text{if } S_W^*(t_1^*+) < 0 \\ S_W^*(t_1^*+) I_{max} + M^*(t_1^*+), & \text{if } S_W^*(t_1^*+) > 0 \end{cases}. \tag{3.8}$$

Since $M(t)$ and $S_W(t)$ are continuous, (3.8) yields

$$S_W^*(t_1^*) = 0. \tag{3.9}$$

This is the switching condition.

When (3.9) is satisfied on some positive time interval $[t_a, t_b]$, the maximum principle fails to provide any information about $I^*(t)$ on this interval. $I^*(t)$ and the corresponding trajectory $x^*(t)$ on such an interval $[t_a, t_b]$ are commonly referred to as a "singular" control and singular arc respectively. Due to the nonlinearity of the system equations (1.2a)-(1.2d), the nonexistence of singular controls cannot be readily established.

By assuming the nonexistence of singular controls, the foregoing results can be summarized by the following theorem.

Theorem 3.1. Suppose (I^*, t_1^*) is an optimal heating program, then it is necessary that

1. there exists an adjoint vector p^* such that

$$\dot{p}^* = \begin{cases} -(\partial H_1^*/\partial x)^T, & \text{on } [t_0, t_1^*) \\ -(\partial H_2^*/\partial x)^T, & \text{on } [t_1^*, t_f] \end{cases}, \tag{3.10}$$

$$p^*(t_f) = [0, p_2^*(t_f), 0, 0]^T, \tag{3.11}$$

$$p^*(t_1^*-) = p^*(t_1^*+), \tag{3.12}$$

where $p_2^*(t_f) = 0$ if $T_i^*(t_f) > T_{id}$ and is unspecified if $T_i^*(t_f) = T_{id}$;

2. $I^*(t) = 0$, on $[t_0, t_1^*)$ and

$$I^*(t) = \begin{cases} I_{min}, & \text{if } S_W^*(t) < 0 \\ I_{max}, & \text{if } S_W^*(t) > 0 \end{cases}, \text{ on } [t_1^*, t_f]; \tag{3.13}$$

3. the following transversality condition is satisfied,

$$S_W^*(t_1^*) = 0. \tag{3.14}$$

The explicit form of (3.10) is given in Appendix G. ||

Proof: Conditions (3.10) and (3.12) follow directly from condition (4.23) of Theorem II.4.1. Condition (3.11) together with the characterization of $p_2^*(t_f)$ can also be derived from this condition. Suppose $T_i^*(t_f) > T_{id}$, then the final condition $T_i^*(t_f) \geq T_{id}$ can be disregarded since it is not active. In other words, our problem can be considered as a free end-point problem. Therefore $p^*(t_f) = 0$ and this implies $p_2^*(t_f) = 0$. On the other hand, if $T_i^*(t_f) = T_{id}$, then the tangent plane Π_f^* to X_f at $x^*(t_f)$ is the hyperplane

$$\Pi_f^* = \{x : \langle x,(0,1,0,0)^T\rangle = T_{id}\}. \tag{3.15}$$

Hence the condition $p^*(t_f) \perp \Pi_f^*$ can be satisfied by any $p_2^*(t_f)$. The other conditions were already derived. ||

Thus, the optimal plasma heating program is of "bang-bang" type such that the neutral injection current takes on either the maximum or minimum value.

Assuming the existence of a unique optimal heating program, we can express the necessary conditions in Theorem 3.1 as a three-point boundary value problem (ThPBVP).

Lemma 3.1. Suppose there exists a unique optimal heating program (I^*,t_1^*). Then x^* and p^* are solutions to the following three-point boundary value problem:

$$\begin{cases} \dot{x}^* = (\partial H_1^*/\partial p)^T \\ \dot{p}^* = -(\partial H_1^*/\partial x)^T \end{cases}, \quad \text{on } [t_0, t_1^*), \tag{3.16}$$

$$\begin{cases} \dot{x}^* = (\partial H_2^*/\partial p)^T \\ \dot{p}^* = -(\partial H_2^*/\partial x)^T \end{cases}, \quad \text{on } (t_1^*, t_f], \tag{3.17}$$

with three-point boundary conditions given by

$$\begin{cases} x^*(t_0) = [T_{e0}, T_{i0}, 0, 0]^T, \\ T_i^*(t_f) \geq T_{id},\ p^*(t_f) = [0, \eta, 0, 0]^T, \\ x^*(t_1^*-) = x^*(t_1^*+),\ p^*(t_1^*-) = p^*(t_1^*+), \\ S_W^*(t_1^*) = 0, \end{cases} \tag{3.18}$$

where $\eta = 0$ if $T_0^*(t_f) > T_{id}$ and η is unspecified if $T_i^*(t_f) = T_{id}$. The optimal injection current $I^*(t)$ satisfies the maximizing condition (3.13). ||

Thus, the solution of the original heating problem can be obtained by solving this ThPVP together with the maximization condition (3.13) of the Hamiltonian H_2 on the second interval $(t_1^*, t_f]$.

5.4. Piecewise Constant Neutral Injection

In actual experiments, it is difficult to realize a continuous variation of the neutral beam current with time. Moreover, it is preferred that the injection current has jumps at prescribed switching times rather than at arbitrary time instants. This leads to the consideration of neutral injection programs such that the beam current is kept constant on each prescribed subinterval of the heating time interval.

Let t_i, $i = 1,2,\ldots,k-1$ be the given switching times such that $t_0 < t_1 < t_2 < \ldots < t_{k-1} < t_f$. We consider a cost functional given by

$$J(I) = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} [A_1 I_p^2 T_e(t)^{-3/2} + REI(t)]dt. \tag{4.1}$$

In this section, it is assumed that the constraint on the magnitude of the beam current $I(t)$ is given by $0 = I_{min} \leq I(t) \leq I_{max}$ on every interval, i.e.,

$$I(t) = I_i \in [0,I_{max}], \text{ on } [t_{i-1},t_i),\ i = 1,2,\ldots,k. \tag{4.2}$$

By assuming $I_{min} = 0$, the subintervals with no injection are simply indicated by $I(t) = 0$.

We formulate the problem of minimizing (4.1) with respect to $I(t)$ as a parameter optimization problem as discussed in Chapter IV. Let $\underline{I}$ be the parameter vector $[I_1,I_2,\ldots,I_k]^T$. The constraint (4.2) is replaced by

$$g_i(\underline{I}) \triangleq I_i(I_i - I_{max}) \leq 0,\ i = 1,2,\ldots,k. \tag{4.3}$$

The requirement that the trajectories reach the target set $\{(T_e,T_i,X_1,X_2) : T_i \geq T_{id}\}$ at the final time t_f can be expressed as

$$h(x_k(\underline{I})) \triangleq T_{id} - \langle\alpha,x(t_f)\rangle \leq 0 \tag{4.4}$$

where $\alpha = [0,1,0,0]^T$, $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{E}^4$ and $x(t_f)$ is the end point of the trajectory x corresponding to the control

$$I(t) = \mathcal{J}(\underline{I}) \triangleq \sum_{i=1}^{k-1} I_i\, \chi_{[t_{i-1},t_i)} + I_k\, \chi_{[t_{k-1},t_f]}. \tag{4.5}$$

Now the plasma heating problem is reformulated as follows:

Problem: Minimize the cost (4.1) with respect to $\underline{I}$ subject to (4.3) and (4.4). ||

We now utilize the results in Chapter IV in deriving necessary conditions for an optimal $\underline{I}^*$. In Chapter IV the right hand side of the system equation was assumed to be continuously differentiable. But $f(x,I)$ does not satisfy this condition since $S_{Di}(T_i)$ is not differentiable in T_i at two points. We need to approximate S_{Di} by some continuously differentiable function of T_i. One possible form of approximation may be a polynomial of T_i. Henceforth we assume that S_{Di} is replaced by its continuously differentiable approximation.

The Hamiltonians H_i on subintervals $[t_{i-1},t_i)$, $i = 1,2,\ldots,k$ have identical forms

$$H_i(x(t),p(t),I_i,\lambda_0) = \overline{S}_W(t)I_i + \overline{M}(t), \quad i = 1,2,\ldots,k, \tag{4.6}$$

where $\overline{S}_W(t)$ and $\overline{M}$ are given by

$$\overline{S}_W(t) = Ep_3(t)T_e^{-3/2}(t) + E^{-1/2}p_4(t) - \lambda_0 RE, \tag{4.7}$$

$$\overline{M} = p_1 f_e + p_2 f_i + (p_3 X_1 + p_4 X_2)(C_1 T_e^{-3/2} + C_2 E^{-3/2})$$

$$- \lambda_0 A_1 I_p^2 T_e^{-3/2}. \tag{4.8}$$

Then for $i = 1,2,\ldots,k$

$$\frac{\partial H_i}{\partial I_i}(x(t),p(t),I_i,\lambda_0) = \overline{S}_W(t) \tag{4.9}$$

does not depend on I_i explicitly.

<u>Theorem 4.1.</u> Suppose $\underline{I}^* = [I_1^*, I_2^*, \ldots, I_k^*]^T$ is optimal, then

1. There exists an adjoint vector p^* which satisfies

$$\begin{cases} \dot{p}^*(t) = -\dfrac{\partial \overline{S}_W^*}{\partial x}(t)\, I^*(t) - \dfrac{\partial \overline{M}^*}{\partial x}(t), \\ p^*(t_f) = [0, -\mu^*, 0, 0]^T, \end{cases} \tag{4.10}$$

where $\mu^* \leq 0$ ($\mu^* = 0$ if $T_i^*(t_f) > T_{id}$), and $I^*(t) = \mathcal{J}(\underline{I}^*)$;

2. Case i) $\displaystyle\int_{t_{i-1}}^{t_i} \overline{S}_W^*(t)dt = 0.$ (4.11)

Then I_i^* is given as a solution of (4.11).

Case ii) $\displaystyle\int_{t_{i-1}}^{t_i} \overline{S}_W^*(t)dt \neq 0,$ then

$$I_i^* = \begin{cases} 0, & \text{when } \int_{t_{i-1}}^{t_i} \overline{S}_W^*(t)dt < 0, \\ I_{max}, & \text{when } \int_{t_{i-1}}^{t_i} \overline{S}_W^*(t)dt > 0. \end{cases} \quad \| \tag{4.12}$$

<u>Proof</u>: The condition (4.10) comes from (IV.3.4) in Theorem IV.3.1 and

$$\frac{\partial H_i^*}{\partial x} = \frac{\partial \overline{S}_W^*}{\partial x} I_i^* + \frac{\partial \overline{M}^*}{\partial x}, \text{ on } [t_{i-1}, t_i),\ i = 1, 2, \ldots, k. \tag{4.13}$$

Condition (4.11) and (4.12) follow from (4.9) and Theorem IV.3.2. ||

It is interesting to compare the optimality condition (4.11) and (4.12) with (3.13) and (3.14) in Section 5.3. We notice that (4.12) "approximates" (3.13) in the sense that as $|t_i - t_{i-1}| \to 0$, the limit of (4.12) coincides with (3.13). In a similar manner, as $|t_i - t_{i-1}| \to 0$

the limit of (4.11) coincides with the switching condition (3.14).

In the next chapter, the two-temperature model of the plasma will be simplified to a single temperature model, for which the optimal heating program can be characterized in greater detail.

CHAPTER VI

SINGLE-TEMPERATURE MODEL OF PLASMA

In this chapter, the minimum injection energy plasma heating problem is discussed using the simplified single-temperature model. The problem of ion temperature stabilization using several forms of feedback control is also discussed.

6.1. Single-temperature Model of Plasma and Minimun Injection Energy Heating

In the recent T.F.R. neutral injection experiments, it was observed that, when neutral injection was introduced after the ion and electron temperatures approached their equilibrium temperatures $\bar{T}_i$ and $\bar{T}_e$ (corresponding to Joule heating only), the electron temperature did not vary significantly during the neutral injection period. This may be explained by the fact that the electrons are in a very high-loss regime when the plasma current I_p is sufficiently large [8].

In such a case, the electron temperature $T_e(t)$ may be approximated by the constant $\bar{T}_e$, and the two-temperature model (V.1.2) may be simplified to a single-temperature model having two equations: one for the ion temperature (T_i) and the other for the injected power transferred to ions (X_2). The relevant equations are:

$$\left.\begin{aligned} \dot{T}_i &= c_1 - h(T_i;I_p) + c_2X_2 \triangleq f_1(T_i,X_2), \\ \dot{X}_2 &= c_3I - c_4X_2 \qquad\qquad\quad \triangleq f_2(X_2,I), \end{aligned}\right\} \tag{1.1}$$

with initial conditions at $t = 0$,

$$T_i(0) = \overline{T}_{i0}, \quad X_2(0) = 0, \tag{1.2}$$

where

$$\left.\begin{aligned} c_1 &= A_3\overline{T}_e^{-3/2}, \quad c_2 = B_5, \\ c_3 &= E^{-1/2}, \quad c_4 = C_1\overline{T}_e^{-3/2} + C_2E^{-3/2}, \\ h(T_i;I_p) &= (A_3\overline{T}_e^{-3/2} + B_4)T_i + S_{Di}(T_i;I_p). \end{aligned}\right\} \tag{1.3}$$

In this chapter, we shall use this model to discuss various problems concerning the plasma heating process.

First, we consider the following minimum input energy plasma heating problem which is a reformulation of the problem discussed in Section 5.1. Throughout this chapter, I_{min} is assumed to be zero. Note that we are employing the multi-stage formulation. The augmented control constraint set $\tilde{\Omega}$ is given by

$$\tilde{\Omega} = \{[I,v_1,v_2]^T : [0,1,0]^T \text{ or } [I,0,1]^T \text{ and } I \in \Omega = [0,I_{max}]\}. \tag{1.4}$$

<u>Problem</u>: Given system (1.1) with initial conditions (1.2); a fixed heating time $[0,t_f]$; and an augmented control constraint set $\tilde{\Omega}$ (1.4). Find an optimal augmented injection program (I^*,v^*) such that $(I^*(t),v^*(t)) \in \tilde{\Omega}$ a.e. on $[0,t_f]$ and the input-energy $J(I,v)$ given by

$$J(I,v) = \int_0^{t_f} [A_1 I_p^2 T_e(t)^{-3/2} + REI(t)v_2(t)]dt, \tag{1.5}$$

is minimized while achieving the desired ion temperature T_{id} at t_f, i.e., $T_i(t_f) \geq T_{id}$. ||

Since we have assumed that $T_e(t)$ is constant, the Joule heating term in (1.5) is constant. Hence for a fixed heating interval $[0,t_f]$, the total energy consumption due to Joule heating is constant. Thus we may eliminate the Joule heating term from the cost functional (1.5) and reduce the above problem to a minimum injection energy problem. Furthermore, since the value $v_2(t)I(t)$ is equal to $I(t)$ a.e. on $[0,t_f]$ under the constraint (1.4), we may identify $v_2(t)I(t)$ with $I(t)$. Consequently, our problem can be simplified to the following standard optimal control problem.

<u>Problem (S)</u>: Given system (1.1) with initial conditions (1.2), a fixed heating time interval $[0,t_f]$, and a control constraint set $\Omega = [0,I_{max}]$. Find an optimal injection program I^* such that $I^*(t) \in \Omega$ a.e. on $[0,t_f]$ and the injection energy $J(I)$ given by

$$J(I) = \int_0^{t_f} REI(t)dt, \tag{1.6}$$

is minimized while achieving the desired ion temperature T_{id} at $t = t_f$, i.e., $T_i(t_f) \geq T_{id}$. ||

For this formulation, we note that the no-injection stage is simply represented by $I(t) = 0$ since $0 \in \Omega$ by the assumption that $I_{min} = 0$. If $I_{min} > 0$, then Ω must be modified to

$$\hat{\Omega} = \{\{0\} \cup [I_{min}, I_{max}]\} \tag{1.7}$$

in order to include the no-injection stage. As we have discussed in Chapter III, a nonconvex constraint set such as $\hat{\Omega}$ causes considerable difficulty in the analysis of optimal solutions.

The existence of an optimal injection heating program I^* for problem (S) can be established with the help of a standard existence theorem presented in Appendix A. In fact, if the set of admissible controls can be shown to be nonempty, then the existence of I^* is guaranteed since all other conditions are satisfied. For this purpose, it is sufficient to assume that $T_i(t_f;I_{max}) \geq T_{id}$ as we did in Section 5.2, where $T_i(t_f;I_{max})$ is the ion temperature at the final time t_f with $I(t) = I_{max}$ for all $t \in [0,t_f]$.

Let $\overline{T}_i(I_{max})$ be the equilibrium ion temperature with $I(t) = I_{max}$ for all t. Henceforth assume that $\overline{T}_i(I_{max}) > T_{id}$, which is obviously required in order to have $T_i(t_f;I_{max}) \geq T_{id}$. Also assume that T_{id} is strictly greater than the equilibrium ion temperature $\overline{T}_i$ corresponding to Joule heating alone. This assumption implies that neutral injection must be introduced for some positive time duration in order to heat the ions to T_{id}.

In the following section, we discuss the characterization of an optimal injection program I^* in detail and give a simple algorithm for the computation of I^*.

6.2. Characterization of Optimal Heating Program

We start with the following result.

Theorem 2.1. The optimal heating program $I^*(t)$ for Problem (S) is given by

$$I^*(t) = \begin{cases} I_{max}, & \text{when } p_2^*(t) > RE^{3/2} \\ 0, & \text{when } p_2^*(t) < RE^{3/2} \end{cases}, \tag{2.1}$$

provided that $p_2^*(t) \not\equiv RE^{3/2}$ on any positive time interval, where p_2^* is the second component of the adjoint vector $p^* = [p_1^*, p_2^*]^T$ satisfying

$$\left.\begin{aligned} \dot{p}^*(t) &= \begin{bmatrix} \dot{p}_1^*(t) \\ \dot{p}_2^*(t) \end{bmatrix} = \begin{bmatrix} \{\partial h(T_i^*(t);I_p)/\partial T_i\} p_1^*(t) \\ -c_2 p_1^*(t) + c_4 p_2^*(t) \end{bmatrix}, \\ p_2^*(t_f) &= 0, \end{aligned}\right\} \tag{2.2}$$

with $\partial h(T_i^*(t);I_p)/\partial T_i \triangleq (\partial h(T_i;I_p)/\partial T_i)|_{T_i^*(t)}$. The values of $I^*(t)$ at the times when $p_2^*(t) = RE^{3/2}$ are determined so as to make I^* continuous from the right hand side. ||

Proof: The Hamiltonian H for Problem (S) is given by

$$H(x,p,I) = -REI + p_1 f_1(T_i, X_2) + p_2 f_2(X_2, I)$$

$$= (c_3 p_2 - RE)I + p_1 f_1(T_i, X_2) - c_4 p_2 X_2. \tag{2.3}$$

Suppose I^* is optimal, then from the maximum principle

1. there exists a nontrivial adjoint vector $p^* = [p_1^*, p_2^*]^T$ which satisfies

$$\dot{p}^*(t) = \begin{bmatrix} -\partial H^*/\partial T_i \\ -\partial H^*/\partial X_2 \end{bmatrix} = \begin{bmatrix} \{\partial h(T_i^*(t);I_p)/\partial T_i\}p_1^*(t) \\ -c_2 p_1^*(t) + c_4 p_2^*(t) \end{bmatrix}, \quad (2.4)$$

$$p^*(t_f) \perp \text{the tangent plane } \Pi_f^* \text{ to } X_f \text{ at } x^*(t_f); \quad (2.5)$$

2. $$H(x^*(t),p^*(t),I^*(t)) = \max_{I\in\Omega} H(x^*(t),p^*(t),I) \text{ a.e. on } [0,t_f]. \quad (2.6)$$

Since $X_f = \{[T_i,X_2]^T : T_i \geq T_{id}\}$, (2.5) leads to the condition $p_2^*(t_f) = 0$. Finally performing the maximization in (2.6) yields (2.1). Note that when $p_2^*(t) = RE^{3/2}$, $I^*(t)$ can take on any value in Ω since the Hamiltonian does not depend on $I^*(t)$. Hence $I^*(t)$ can be chosen so as to make I^* continuous from the right hand side. ||

When $p_2^*(t) = RE^{3/2}$ for some positive time interval, the maximum principle fails to provide any information about $I^*(t)$ on that interval. $I^*(t)$ on such an interval becomes a "singular" control and some higher order optimality condition must be employed in order to specify $I^*(t)$. However, it will be shown in the next lemma that no singular interval exists.

<u>Lemma 2.1.</u> p_1^* is strictly monotone in t and p_2^* can not be equal to $RE^{3/2}$ for any positive time duration. ||

<u>Proof:</u> First we show that $p_{10}^* \neq 0$. Suppose $p_{10}^* = 0$, then from (2.2), $p_1^*(t) \equiv 0$ on T, and hence $p_2^*(t) \equiv 0$ on T. This contradicts the fact that p^* is nonzero. Now from (1.3) and the definition of $S_{Di}(T_i;I_p)$,

$$\frac{\partial h}{\partial T_i}(T_i;I_p) = \left\{\begin{array}{ll} G + B_1 I_p^{-2} T_i^{-1/2}/2, & \text{for } 0 \le T_i < T_{i1} \\ G + 5B_2 I_p^{-1} T_i^{3/2}/2, & \text{for } T_{i1} < T_i < T_{i2} \\ G + B_3 I_p^{-2} T_i^{-1/2}/2, & \text{for } T_{i2} < T_i \end{array}\right\}, \qquad (2.7)$$

where $G = A_3 \overline{T}_e^{-3/2} + B_4$, $T_{i1} = (B_1/(B_2 I_p))^{1/2}$ and $T_{i2} = (B_3/(B_2 I_p))^{1/2}$. Obviously $\partial h(T_i;I_p)/\partial T_i > G$ for all $T_i \ge 0$ except at T_{i1} and T_{i2}. Consequently, for $T_i^* = T_i^*(t)$ such that $\partial h(T_i^*(t);I_p)/\partial T_i$ exists a.e. on T, $p_1^*(t)$ is strictly monotone. Now suppose that p_2^* is constant on some interval $[t_1,t_2] \in T$, $t_1 \neq t_2$. *Then from* (2.4), $p_1^*(t) = c_4 p_2^*(t)/c_2 \equiv$ constant since $\dot{p}_2^*(t) = 0$ on $[t_1,t_2]$. This contradicts the strict monotonicity of p_1^*. ‖

Thus we have established the nonexistence of a singular interval. In the next lemma, we find the maximum number of roots for $p_2(t) = RE^{3/2}$.

Lemma 2.2. The equation

$$p_2(t) = RE^{3/2}, \qquad (2.8)$$

can have at most two roots on $T = [0,t_f]$. ‖

Proof: The solutions to (2.2) on $[0,t_f]$ have the forms

$$\left.\begin{array}{l} p_1(t) = p_{10}\exp\{\int_0^t [\partial h(T_1^*(s);I_p)/\partial T_i]ds\} \\ p_2(t) = \int_t^{t_f} \exp\{c_4(t-s)\}c_2 p_1(s)ds \end{array}\right\}, \qquad (2.9)$$

where p_{10} is an unspecified constant. When $p_{10} < 0$, it follows from (2.9) that $p_1(t)$ **and** $p_2(t)$ are both negative for all $t \in T$. Hence (2.8) has no solution. Now, assume that $p_{10} > 0$. Let $\bar{t} \in (0,t_f)$ be a stationary point of p_2, i.e., $\dot{p}_2(\bar{t}) = 0$. Then, we have from (2.2) and Lemma 2.1,

$$\ddot{p}_2(\bar{t}) = -c_2\dot{p}_1(\bar{t}) + c_4\dot{p}_2(\bar{t}) = -c_2\dot{p}_1(\bar{t}) < 0. \tag{2.10}$$

This shows that any stationary point $\bar{t}$ is a local maximum point of p_2. Since p_2 does not have a corner which can be a local minimum, p_2 can have at most one local maximum. By Lemma 2.1, p_2 cannot be a constant on a positive time duration. Thus there exists at most one relative maximum point of p_2 in $(0,t_f)$. Consequently, there exists at most two roots for (2.8) on T. $\|$

Next we consider a special property of the trajectory with $I(t) = I_{max}$ for all $t \in T$.

<u>Lemma 2.3</u>. Let $x(\cdot;I_{max}) \triangleq [T_i(\cdot;I_{max}),X_2(\cdot;I_{max})]^T$ be a trajectory corresponding to the maximum injection $I(t) = I_{max}$ for all $t \in T$. Then $T_i(t;I_{max})$ is the highest attainable ion temperature at any time $t \in (0,t_f]$. $\|$

<u>Proof:</u> Suppose there exists a trajectory $\tilde{x} = [\tilde{T}_i,\tilde{X}_2]^T$ corresponding to an injection program $\tilde{I}$, such that $\tilde{T}_i(t_1) > T_i(t_1;I_{max})$ for some time $t_1 \in (0,t_f]$. Since $\tilde{T}_i(t)$ and $T_i(t;I_{max})$ are continuous there exists a time t_2, $0 \le t_2 < t_1$, such that $\tilde{T}_i(t_2) = T_i(t_2;I_{max})$

and $\tilde{T}_i(t) > T_i(t;I_{max})$ for all $t \in (t_2,t_1]$. Then we must have $\dot{\tilde{T}}_i(t_2) > \dot{T}_i(t_2;I_{max})$. Now, from (1.1),

$$\dot{\tilde{T}}_i(t_2) - \dot{T}_i(t_2;I_{max}) = c_2\{\tilde{X}_2(t_2) - X_2(t_2;I_{max})\}. \tag{2.11}$$

Also from (1.1) and (1.2), for any injection program I,

$$X_2(t) = \int_0^t c_3 \exp\{-c_2(t - s)\}I(s)ds. \tag{2.12}$$

From (2.11) and (2.12),

$$\dot{\tilde{T}}_i(t_2) - \dot{T}_i(t_2;I_{max}) = c_2 \int_0^{t_2} c_3 \exp\{-c_4(t_2 - s)\}\{\tilde{I}(s) - I_{max}\}ds. \tag{2.13}$$

Here, since $\tilde{I}$ satisfies the constraint $I(t) \in \Omega = [0,I_{max}]$ a.e. on T, we must have $\tilde{I}(s) - I_{max} \leq 0$ a.e. on $[0,t_2]$. Consequently $\dot{\tilde{T}}_i(t_2) - \dot{T}_i(t_2;I_{max}) \leq 0$. This is a contradiction. ||

Now we are ready to present three possible forms for an optimal heating program.

<u>Theorem 2.2</u>. The optimal neutral injection heating program takes on one of the following three possible forms:

$$I^*(t) = I_{max} \quad \text{on} \quad [0,t_f], \tag{2.13}$$

$$I^*(t) = \begin{Bmatrix} I_{max} & \text{on} & [0,t_{s1}) \\ 0 & \text{on} & [t_{s1},t_f] \end{Bmatrix}, \tag{2.14}$$

$$I^*(t) = \begin{Bmatrix} 0 & \text{on} & [0,t_0) \\ I_{max} & \text{on} & [t_0,t_{s2}) \\ 0 & \text{on} & [t_{s2},t_f] \end{Bmatrix}. \tag{2.15}$$

These programs are illustrated in Figure 6.1. $\|$

Proof: From Lemmas 2.1 and 2.2, $p_2(t)$ can only assume one of the following three forms:

$$1. \quad p_2(t) < q \quad \text{for all} \quad t \in T; \tag{2.16}$$

$$2. \quad \left\{\begin{array}{ll} q < p_2(t) & \text{for} \quad [0,t_{s1}) \\ 0 \leq p_2(t) < q & \text{for} \quad (t_{s1},t_f] \end{array}\right\} ; \tag{2.17}$$

$$3. \quad \left\{\begin{array}{ll} 0 < p_2(t) < q & \text{for} \quad [0,t_0) \\ q < p_2(t) & \text{for} \quad (t_0,t_{s1}) \\ 0 \leq p_2(t) < q & \text{for} \quad (t_{s1},t_f] \end{array}\right\} ; \tag{2.18}$$

where $q = RE^{3/2}$ and t_{s1}, t_{s2} and t_0 are solutions for (2.8). By Theorem 2.1, the cases (2) and (3) lead to the injection program (2.14) and (2.15) respectively. Case (1) corresponds to $I^*(t) = 0$ for all $t \in T$, and is not relevant for our problem since $T_{id} > \overline{T}_i$. The heating program (2.15) corresponds to the special case $T_i(t_f;I_{max}) = T_{id}$. For this case, by Lemma 2.3, there exists only one trajectory $x(\cdot;I_{max})$ which satisfies the final condition $T_i(t_f) \geq T_{id}$. Therefore $x(\cdot;I_{max})$ is optimal. $\|$

Thus our remaining task is to give a condition for determining the form of I^* for each particular case. For this purpose, we look in detail at the various trajectories of T_i in the time domain. In Figure 6.2:

1. The curve C_1 corresponds to the trajectory with $I(t) = I_{max}$ for all $t \geq 0$. C_1 intersects the line

$T_i = T_{id}$ at time t_{cl} and approaches the equilibrium ion temperature $\bar{T}_i(I_{max})$ as $t \to \infty$. (The asymptotic stability, in the first quadrant Q^+ of the T_i-X_2 plane, of an equilibrium point $\bar{x}(I_{max}) = (\bar{T}_i(I_{max}), \bar{X}_2(I_{max}))$ corresponding to $I(t) = I_{max}$ for all $t \geq 0$ can be established by a method which will be used in Theorem 3.1 in the next section.)

2. The curve C_2 corresponds to the injection program (2.14) for some $t_{sl} > 0$. After the neutral injection is cut at t_{sl}, the value of $X_2(t)$ decays exponentially. Hence $T_i(t)$ continues to rise until it reaches a maximum value at some time $t_m > T_{sl}$, and then decays monotonically.

3. The curve C_3 corresponds to the special case where the maximum value $T_i(t_m)$ is equal to T_{id}. For this special case, the injection cut-off time t_{sl} and the time when $T_i(t)$ reaches T_{id} are denoted by t_{cs} and t_{cm} respectively.

We recognize that t_{cl}, t_{cs} and t_{cm} are known once I_{max} is given. This suggests that we may determine a special form for $I^*(t)$ by comparing t_f with these times. In fact, once t_{cl}, t_{cs} and t_{cm} or equivalently the curves C_1 and C_3 are computed, a special form of I^* for a particular problem can be found by comparing t_f to t_{cl} and t_{cm}.

We conclude this section by summarizing the preceding discussion in the following theorem.

<u>Theorem 2.3</u> (Algorithm). A special form of I^* for a given t_f can be computed with the following procedure.

1. Find t_{cl}, t_{cs} and t_{cm} by computing curves C_1 and C_3 for a given I_{max}.
2. Compare t_f with t_{cl} and t_{cm}.
3. If $t_f < t_{cl}$, then conclude that no solution exists.
4. If $t_f = t_{cl}$, then set $I^*(t) = I_{max}$ for all $t \in [0,t_f]$.
5. If $t_{cl} < t_f \le t_{cm}$, then set

$$I^*(t) = \left\{ \begin{array}{ll} I_{max}, & \text{on } [0,t_s^*) \\ 0\ , & \text{on } [t_s^*,t_f] \end{array} \right\} , \qquad (2.19)$$

   where t_s^* is computed using the condition $T_i(t_f) = T_{id}$.
6. Otherwise, set

$$I^*(t) = \left\{ \begin{array}{ll} 0\ , & \text{on } [0,t_f - t_{cm}) \\ I_{max}, & \text{on } [t_f-t_{cm},t_f-t_{cm}+t_{cs}) \\ 0\ , & \text{on } [t_f-t_{cm}+t_{cs},t_f] \end{array} \right\} . \| \qquad (2.20)$$

<u>Proof</u>: When $t_f < t_{cl}$, Problem (S) has no solution since T_{id} cannot be reached. If $t_f = t_{cl}$, then there exists only one trajectory with $T_i(t_f) = T_{id}$, and it is realized by $I(t) = I_{max}$ for all t. Hence I^* takes on the form (a) in Fig. 6.1. When $t_{cl} < t_f \le t_{cm}$, I^* is of the form (b) in Fig. 6.1, and the cut-off time t_s can be computed from the condition $T_i(t_f) = T_{id}$. Finally, when $t_f > t_{cm}$, I^* assumes the form (c) in Fig. 6.1. For this case, the initialization of the neutral injection is delayed until $t_0 = t_f - t_{cm}$ and the injection is cut at $t_s = t_0 + t_{cs}$. $\|$

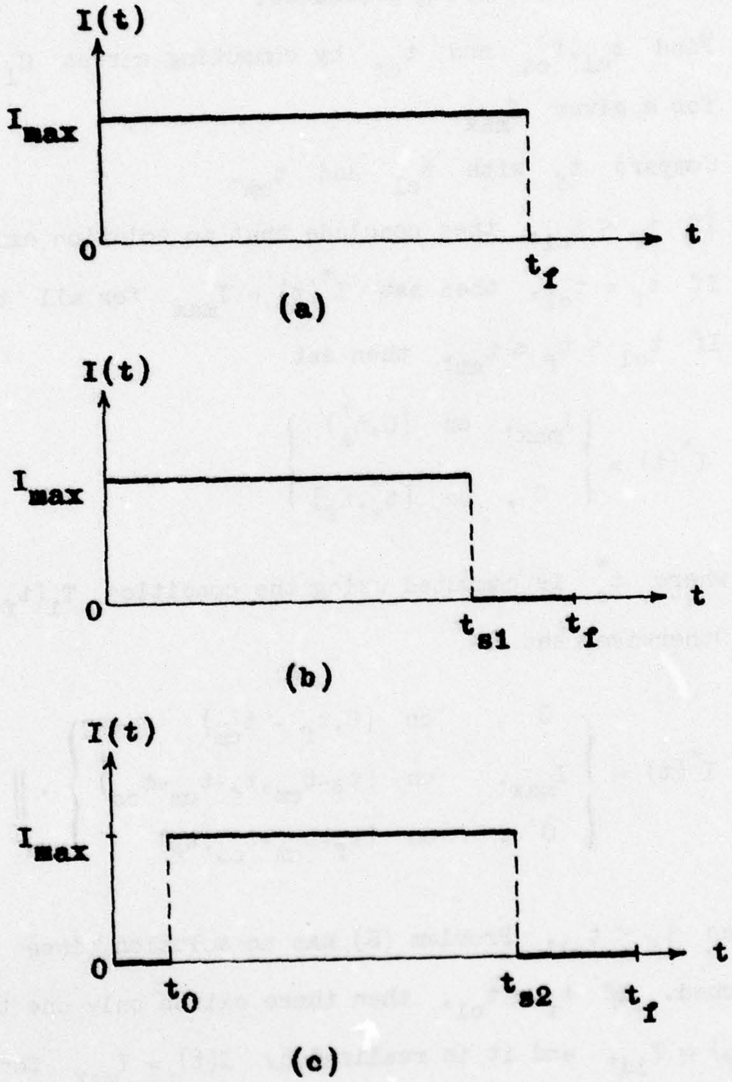

Fig. 6.1. Possible Forms for I^*

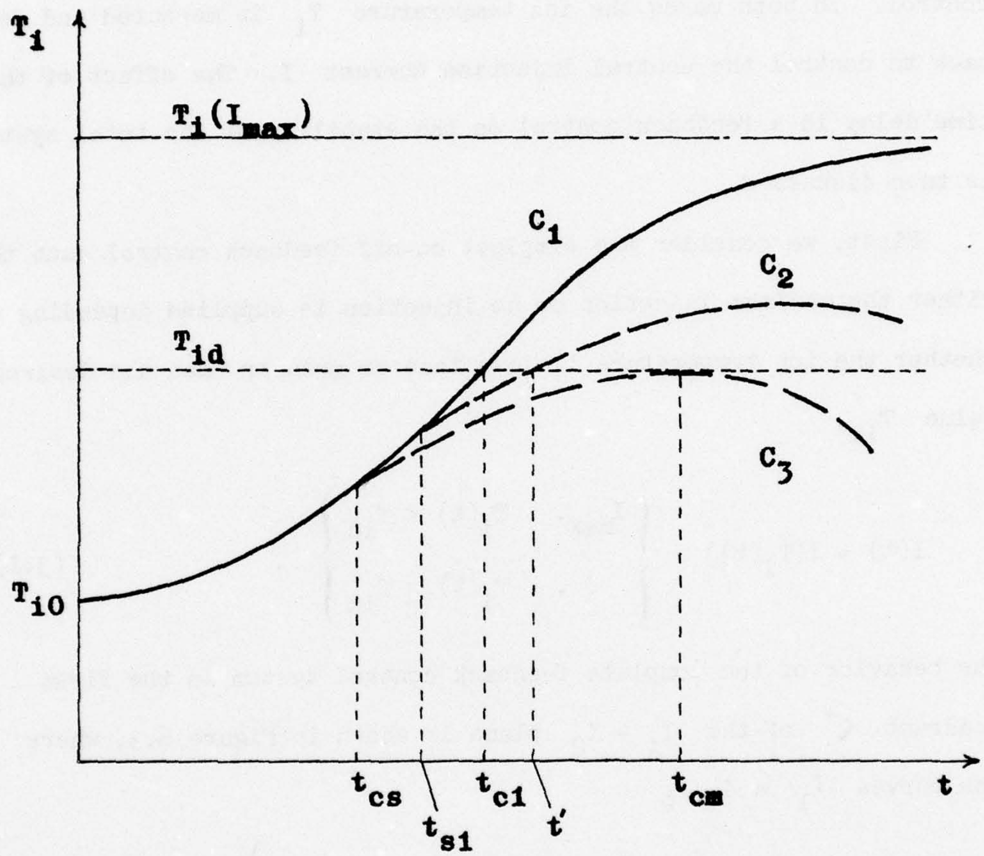

Fig. 6.2. Ion-Temperature Trajectories in Time Domain

6.3. Feedback Controls and Stability

In this section, we discuss the regulation of the ion temperature about its desired value. We consider two different forms of feedback control. In both cases the ion temperature T_i is measured and fed back to control the neutral injection current I. The effect of the time delay in a feedback control on the stability of the total system is then discussed.

First, we consider the simplest on-off feedback control such that either the maximum injection or no injection is supplied depending on whether the ion temperature is less than or greater than the desired value T_{id},

$$I(t) = I(T_i(t)) = \begin{Bmatrix} I_{max}, & T_i(t) < T_{id} \\ 0\ , & T_i(t) \geq T_{id} \end{Bmatrix} . \tag{3.1}$$

The behavior of the complete feedback control system in the first quadrant Q^+ of the T_i - X_2 plane is shown in Figure 6.3, where the curves Γ_1 and Γ_2

$$\left.\begin{aligned} \Gamma_1 &: X_2 = \{h(T_i;I_p) - c_1\}/c_2, \\ \Gamma_2 &: X_2 = c_3 I(T_i)/c_4, \end{aligned}\right\} \tag{3.2}$$

are the zero-rate curves for T_i and X_2 respectively. Since Γ_2 is discontinuous, Γ_1 and Γ_2 may not intersect and the equilibrium point (e.p.) may not exist. In fact, when T_{id} is less than the equilibrium ion temperature corresponding to $I(t) = I_{max}$ as is the case shown in Fig. 6.3, the curves Γ_1 and Γ_2 do not intersect.

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Hence there is no e.p. in the usual sense. But the nature of the directional field indicates that the trajectory emerging from any point in Q^+ approaches the neighborhood of a point $\bar{x}$ shown in Fig. 6.3. The results of simulation show that the trajectories approach $\bar{x}$ in a spiral manner and endorses the above observation. A curve F_1 in Fig. 6.3 is a typical result of simulation.

As the trajectory approaches T_{id}, the control $I(t)$ oscillates rapidly between I_{max} and 0. This oscillation can be avoided by replacing the discontinuous feedback control (3.1) by a continuous feedback control. A possible form is given by

$$I(T_i(t)) = \left\{ \begin{array}{ll} I_{max}[1 - \exp\{-(T_{id}-T_i(t))^2/\sigma\}], & T_i(t) < T_{id} \\ 0 & , T_{id} \leq T_i(t) \end{array} \right\}, \quad (3.3)$$

where $\sigma > 0$ is a constant.

A typical trajectory with this feedback control is shown by F_2 in Figure 6.4.

Now the stability of an e.p. x_e, given by the intersection of Γ_1 and Γ_2, is established by considering the divergence of the velocity vector and applying Bendixson's theorem.

Theorem 3.1. The e.p. x_e in Fig. 6.4 is asymptotically globally stable in the first quadrant Q^+ of $T_i - X_2$ plane. ||

Proof: The divergence of the velocity vector $f \triangleq (f_1, f_2)$

$$\text{div } f = (\partial f_1/\partial T_i) + (\partial f_2/\partial X_2) = -\partial h(T_i;I_p)/\partial T_i - c_4, \quad (3.4)$$

is nonpositive for almost all (T_1, X_2) in Q^+ since h is strictly monotone increasing and is differentiable except at the two points T_{i1} and T_{i2} as defined earlier. Hence by Bendixson's Theorem ([7],[20]) no periodic solutions or limit cycles can exist in Q^+. Now consider a rectangular region D illustrated in Fig. 6.4, positioned such that two sides are on the T_1 and X_2 axes and the corner opposite the origin is on the curve Γ_1. D also contains the nonzero section of Γ_2 given by (3.2) and (3.3). The directional field indicates that the trajectories are either directed into the interior of D or along its boundary. Since there are no limit cycles in D, the trajectories must converge to x_e as $t \to \infty$. Furthermore, since D can be enlarged to completely cover Q^+, x_e is globally asymptotically stable in Q^+. ||

In an actual ion temperature control system, the effect of time-delays due to the neutral-particle transit-time in the injector and the data processing time associated with ion-temperature measurement should be considered. When the energy of the injected particles is in the KeV range, the transit-time delay is negligible as compared to the characteristic time associated with the energy exchange between the ions and the injected particles. When the ion temperature measurement time delay is present, the feedback control takes the form

$$I(t) = I_{max}\, q(T_{id} - T_1(t - \tau_d)), \tag{3.5}$$

where q is assumed to be a differentiable function of its argument and τ_d is the time-delay. The effect of τ_d on stability can be

estimated by considering the equations of the feedback system (1.1), (3.5) linearized about its equilibrium point $x_e = (T_{ie}, X_{2e})$:

$$\left.\begin{aligned} d\delta T_i(t)/dt &= \gamma\delta T_i(t) + c_2\delta X_2(t), \\ d\delta X_2(t)/dt &= c_3\Delta\delta T_i(t - \tau_d) - c_4\delta X_2(t), \end{aligned}\right\} \qquad (3.6)$$

where

$$\left.\begin{aligned} \delta T_i &= T_i - T_{ie}, \quad \delta X_2 = X_2 - X_{2e}, \\ \gamma &= -(\partial h(T_o;I_p)/\partial T_i)|_{T_{ie}}, \\ \Delta &= -I_{max}(\partial g(\xi)/\partial\xi)|_{(T_{id}-T_{ie})}. \end{aligned}\right\} \qquad (3.7)$$

The stability of x_e or the stability of the trivial solution of (3.6) is ensured when the characteristic equation for (3.6) given by

$$z^2 + (c_4-\gamma)z - c_4\gamma - c_2c_3\Delta\exp(-\tau_d z) = 0, \qquad (3.8)$$

has no root with a positive real part. The stability boundaries in the (Δ,τ_d) parameter plane can be found by setting $z = 0$ and $z = iy$ in (3.8) where y is real and $i = \sqrt{-1}$. This leads to

$$\Delta = (c_4\gamma)/(c_2c_3), \qquad (3.9)$$

$$\left.\begin{aligned} y^2 + c_4\gamma + c_2c_3\Delta\cos(\tau_d y) &= 0, \\ (c_4-\gamma)y + c_2c_3\Delta\sin(\tau_d y) &= 0. \end{aligned}\right\} \qquad (3.10)$$

By solving (3.10) for Δ and τ_d in terms of y, the stability region is given by

$$
\left.\begin{aligned}
&-[(y^2+c_4^2)(y^2+\gamma^2)]^{1/2}/(c_2c_3) \le \Delta \le -(c_4\gamma)/(c_2c_3), \\
&\tau_d \le \frac{1}{y}\tan^{-1}\{y(c_4-\gamma)/(y^2+c_4\gamma)\} .
\end{aligned}\right\} \qquad (3.11)
$$

This stability region corresponding to actual parameter values is shown in Figure 6.5. It is evident that the maximum allowable time-delay τ_d is reduces as Δ becomes more negative.

Thus we have shown that stability can be achieved by a suitable choice of parameters even in the presence of measurement time-delay. However, since the derived stability region in Fig. 6.5 is for the linearized system (3.6), it is only applicable to the original system when the state is sufficiently close to an equilibrium point. In an actual ion temperature regulation system, we may use two modes of operation: (1) apply the minimum energy heating program to steer the system state to some neighborhood of an equilibrium point x_e, and then (2) switch to the feedback control (3.3) for regulating the ion temperature about x_e. Note that the equilibrium ion temperature T_{ie} is lower than T_{id} when the smooth feedback control (3.3) is used. This can be avoided simply by replacing T_{id} in (3.3) with some $\tilde{T}_{id} > T_{id}$ so that the curves Γ_1 and Γ_2 intersect when $T_i = T_{id}$.

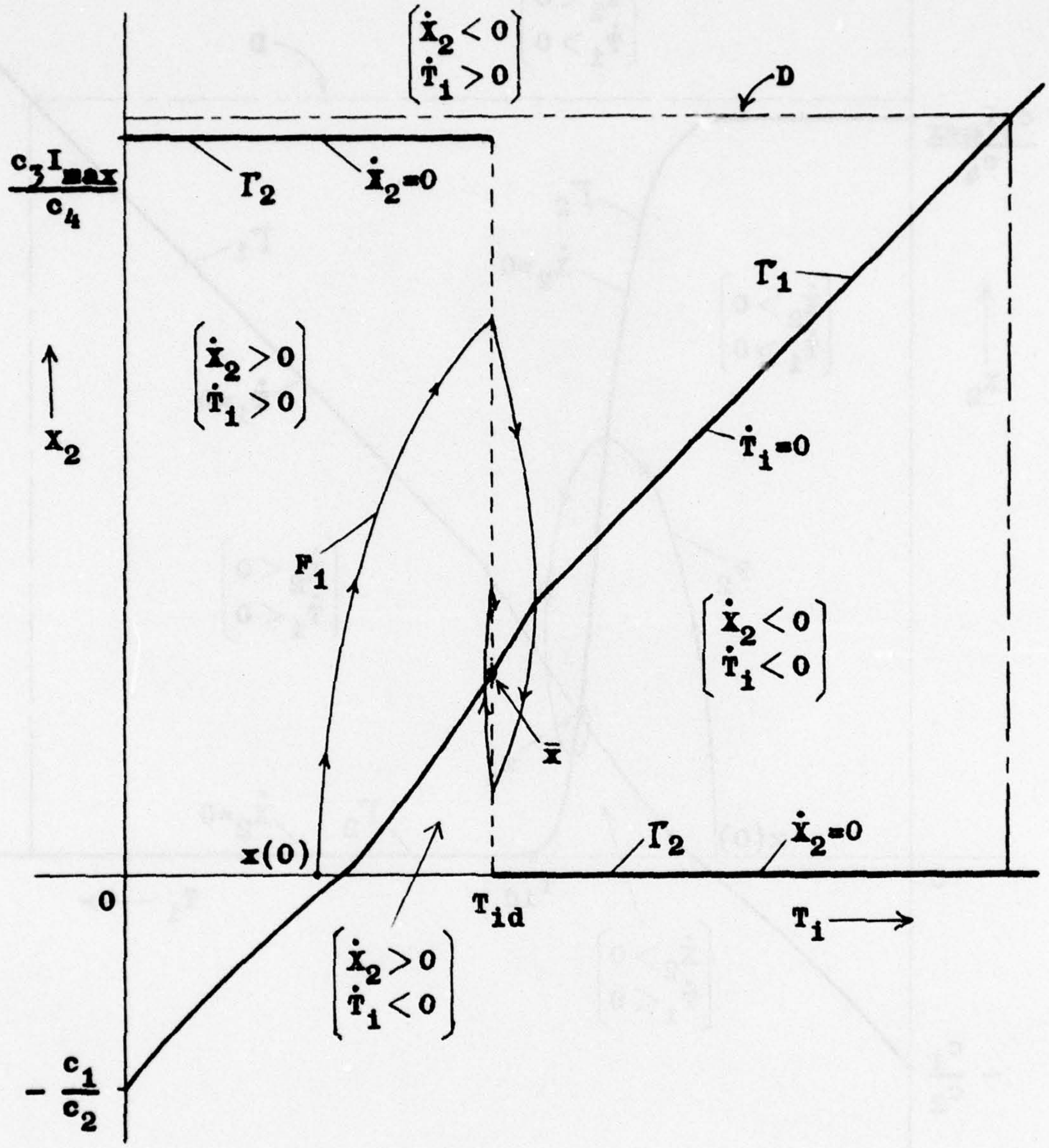

Fig. 6.3. Typical Trajectory of the Relay Feedback Controlled Injection System

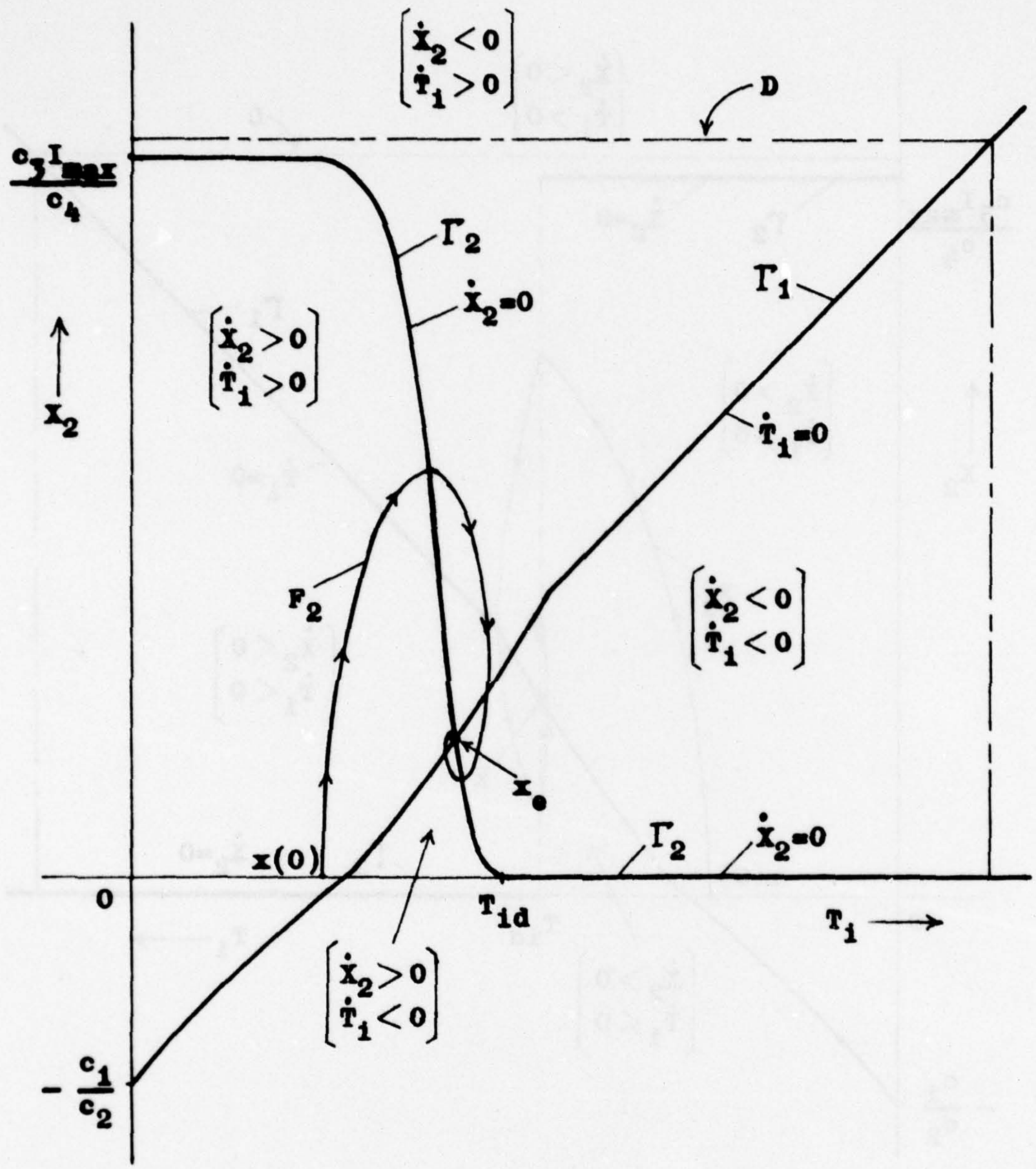

Fig. 6.4. Typical Trajectory of the Smooth Feedback Controlled Injection System

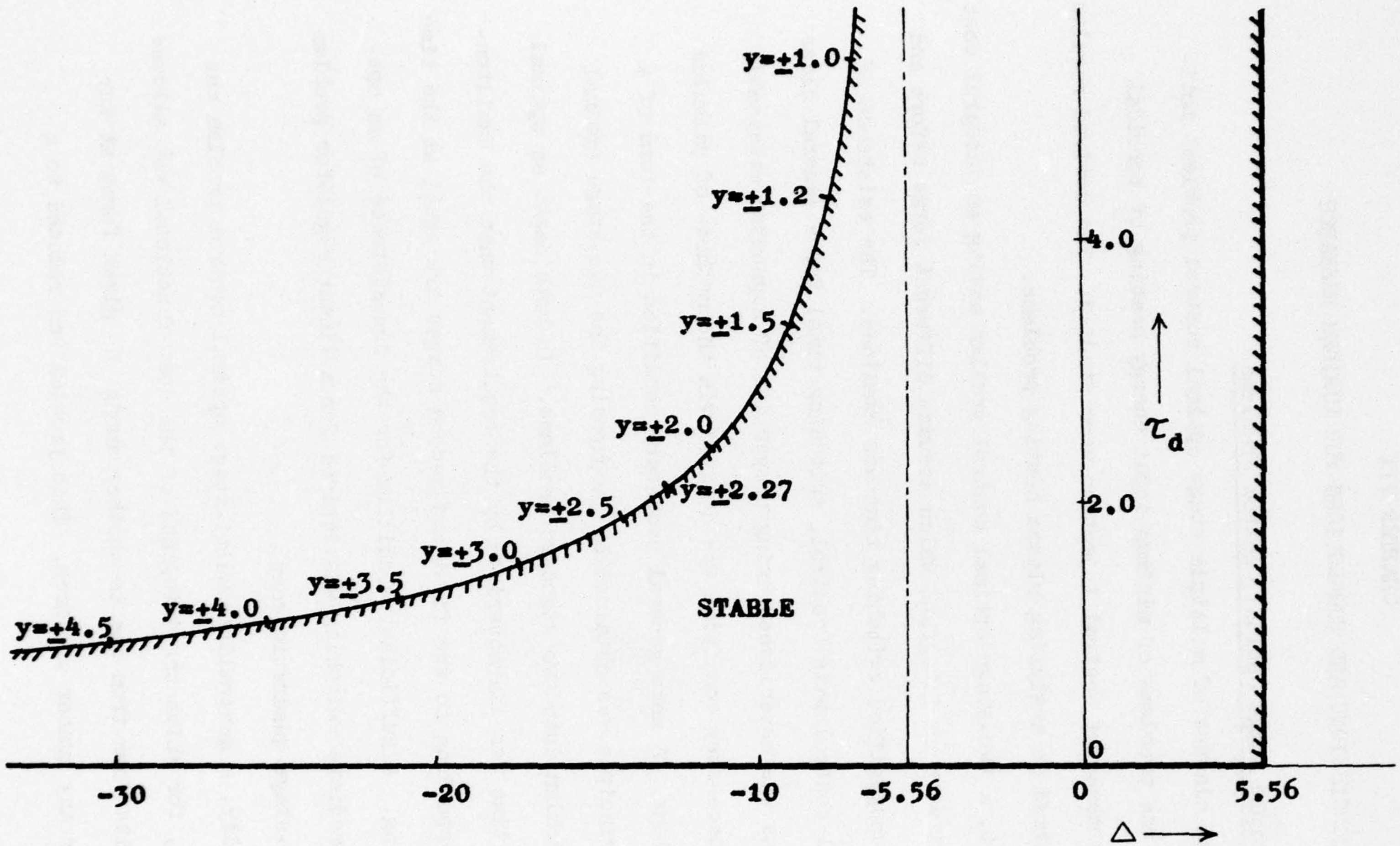

Fig. 6.5. Stability Region in Δ(Gain) - τ_d(Time-Delay) Space

CHAPTER VII

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

7.1 Multiple Stage Optimal Control Problems

Three classes of multiple stage optimal control problems motivated by the problems of minimum input energy heating of toroidal plasma by means of neutral injection were studied. The derived results were utilized in analyzing plasma heating problems.

First, a two-stage optimal control problem having an integral cost functional with an integrand which assumed different forms before and after an unspecified switching time was examined. The existence of an optimal control pair (control, switching time) for a general class of problems was established using properties of augmented attainable sets. A necessary condition was derived via the methods of calculus of variations. A more general necessary condition in the form of a maximum principle was obtained by decomposing the two-stage optimal control problem into two standard problems. In both cases an optimal switching time was characterized by the requirement that the Hamiltonians corresponding to the first and second stages are equal at the time of switching. A sufficient condition for the nonexistence of an optimal intermediate switching was derived for a linear regulator problem with a two-stage quadratic cost.

Secondly, a generalized multi-stage optimal control problem was considered, for which the integrand of the cost functional was allowed to change its form from one to another among N given forms at any instant for any number of times. This problem was reduced to a

standard optimal control problem by introducing a set of auxiliary controls. It was shown that finding an optimal switching schedule is equivalent to finding an optimal auxiliary control. The investigation of the existence of an optimal auxiliary control showed the possibility of chattering controls due to the fact that the augmented control constraint set is never convex.

Thirdly, to comply with the engineering requirement that the neutral injection program be piecewise constant, we considered a fixed k-stage optimal control problem with a cost functional assuming k preassigned forms on k fixed subintervals. The control was assumed to be constant on each subinterval. It was shown that this problem can be formulated as a parameter optimization problem in k-dimensional Euclidean space E^k by recognizing the *one-to-one correspondence* between vectors in E^k and the class of control functions under consideration. The reformulation was performed without any approximation and a necessary condition for an optimal vector was derived in integral form.

In regard to the general two-stage problem, it is important to find a condition which guarantees the nonexistence of an optimal intermediate switching as was derived for a special case in Section 2.5. It was discussed in Section 2.3 that the optimal control pair is generally obtained by solving a three-point boundary value problem, whereas, if it is known a priori that the optimal switching time is one of the end times, then only a two-point boundary value problem needs to be solved.

From an engineering point of view, the existence of an optimal measurable control is not sufficient for the multi-stage problem. The existence of an optimal piecewise continuous control is essential for

the multi-stage problem since it guarantees the finiteness of the optimal number of switchings. In general, certain additional regularity assumptions are required for the existence of an optimal piecewise continuous control. For example, piecewise analyticity of the right hand side of a system equation was assumed in [21] so as to insure the existence of an optimal piecewise continuous control for the case of linear systems. Problems with more general systems are still remaining as subjects for further research.

Another interesting variant of the multi-stage problem would be to consider the case where the maximum number of switchings is fixed. For this case, the existence of an optimal solution may be shown by utilizing the finiteness of the maximum number of switchings. But it appeared to be very difficult to characterize an optimal solution under such conditions as this, since a fixed maximum number of switchings implies a fixed maximum number of discontinuities of auxiliary controls.

It should be noted that we could have allowed the system equation as well as the cost functional to change its form at the switching times. The extension of the derived results to this case may be carried out without difficulty.

The multiple stage optimal control problems considered in this dissertation are very general and are applicable not only to plasma heating problems but also to other practical problems. For example, the problems with abrupt change(s) in the cost criterion and/or system equation at unknown time(s) may be formulated as either a two-stage or a

multi-stage optimal control problem and may be solved utilizing the results presented in this dissertation.

7.2 Plasma Heating Problem

The problem of minimum input energy plasma heating by means of neutral injection was formulated as a two-stage optimal control problem using a two-temperature model of the plasma. This mathematical optimization problem was shown to be meaningful by showing the existence of an optimal solution. The derived optimal heating problem is of "on-off" type so that the implementation would not cause any serious difficulties. In case physical conditions require the switching of neutral beam current at certain time instants, we can adopt the optimal piecewise constant injection program discussed in Section 5.4.

Given the present technology the energy consumption due to neutral injection can only amount to a small percentage of that due to Joule heating. However, in the future large machines such as the J.E.T. (Joint European Tokamak) neutral injection will be the primal source of plasma heating and its energy consumption will become a dominant part of the total input energy. In such cases, the heating system is required to operate at levels of high efficiency so that energy extraction from the plasma is possible. Therefore, the optimal heating problem considered here will become very important since employment of the optimal heating program will result in a significant conservation of the input energy.

The detailed characterization of an optimal heating program was carried out using the simplified single-temperature model of the plasma.

It was shown that the optimal neutral injection program takes on one of the three possible forms, all of which are of "on-off" type. The special forms for individual cases can be determined simply by comparing the heating time duration to the two critical times which can easily be computed once all the parameters are given.

The investigation of feedback regulation of ion temperature showed that stable equilibrium ion temperatures can be achieved for several types of feedback controls even in the presence of ion temperature measurement time-delay. This suggests the possibility of the automatic ion temperature regulation using feedback-controlled neutral injection.

It should be noted that the two-temperature model of the plasma is a highly simplified model and there are several limitations. It would be important in future studies to include the dynamics of the neutral beam injector in this model. As the magnitude of neutral injection becomes large, the underlying physical characteristics of the injector become important factors in considering efficient plasma heating. In fact, it is known that as the kinetic-energy of the accelerated ions becomes large, the efficiently of the neutralizer drops significantly. Also in the future large machines, the kinetic-energy of the neutral particles will be required to be much larger than that in the existing machines, since the penetration depth of the neutral particles into the plasma must be greater. The other limitation of the two-temperature model is that the characteristics of the plasmas related to their spatial temperature profiles are not revealed using this model. For example, it has been proposed that the transition of the

electron diffusion law from one regime to another depends on the spatial electron temperature gradient.+

Finally, it would be interesting to consider the dynamics of the electron and ion densities since the plasma density is one of the three important parameters characterizing the nuclear fusion reaction. The problem of obtaining the maximum ion density by means of neutral injection would be an important and interesting problem to consider.

+Personal conversation with M. Cotsaftis.

APPENDIX A

STANDARD EXISTENCE THEOREM

Consider the following optimal control problem:

Problem. Given a system $\dot{x} = f(x,u,t)$, a control time interval $[t_0,t_f]$, terminal conditions $x(t_0) \in X_0$ and $x(t_f) \in X_f$ where X_0 and X_f are nonempty closed sets in $\mathbb{E}^n$, and a control constraint set $\Omega(t) \in \mathbb{E}^n$, $t \in [t_0,t_f]$, find an admissible control u^* which minimizes the cost functional

$$J(u) = \int_{t_0}^{t_f} L(x(t),u(t),t)dt, \tag{A.1}$$

i.e., $J(u^*) \leq J(u)$ for any admissible control u. (A control u is said to be admissible if (1) $u(t) \in \Omega(t)$ a.e. on $[t_0,t_f]$, and (2) the corresponding trajectory φ satisfies $\varphi(t_0) \in X_0$ and $\varphi(t_f) \in X_f$.) ||

We state the following Lemma without proof. (For a proof, refer to [3].)

Lemma A.1. Assume that the following hypotheses are satisfied.

1. The set of admissible controls Λ is nonempty;
2. There exists a compact set $\mathcal{R}_0 \subset \mathbb{E}^{n+1}$ such that for all admissible trajectories φ, $(t,\varphi(t)) \in \mathcal{R}_0$, for all $t \in [t_0,t_f]$;
3. at any $\tau \in [t_0,t_f]$, for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for $t \in [t_0,t_f]$

$|t-\tau| < \delta(\varepsilon)$ implies $\Omega(t)$ is contained in a closed ε-neighborhood of $\Omega(\tau)$;

4. the set $\Omega(t)$ is compact for all $t \in [t_0,t_f]$;

5. for each $(t,x) \in \mathcal{R}$, the set $v^+(t,x)$ is convex, where

$$v^+(t,x) = \{\hat{y} = (y_0,y) \in \mathbb{E}^{n+1} : y_0 \geq L(x,u,t),$$
$$y = f(x,u,t),\ u \in \Omega(t)\}, \tag{A.1}$$

and $\mathcal{R} \subseteq \mathbb{E}^{n+1}$ is an open set such that $\mathcal{R}_0 \subseteq \mathcal{R}$;

6. L is lower semi-continuous and f is continuous in $\mathcal{R} \times U$, where $U \subseteq \mathbb{E}^m$ is an open set such that $\bigcup_{t\in[t_0,t_f]}\Omega(t) \subseteq U$.

Then there exists an optimal control u^* such that

$$J(u^*) \leq J(u)$$

for all admissible u. ||

APPENDIX B

COMPUTATION OF FIRST-ORDER VARIATIONS

Let (u^*,t_1^*) and x^* be an optimal control pair and a corresponding optimal trajectory. Let $(\hat{u},\hat{t}_1)$ and $\hat{x}$ be a perturbed control pair and a corresponding trajectory such that,

$$(\hat{u},\hat{t}_1) = (u^* + \varepsilon v, t_1^* + \varepsilon\tau), \tag{B.1}$$

$$\hat{x} = x^* + \varepsilon y, \tag{B.2}$$

where $\varepsilon > 0$ and (v,τ) and y are respectively a perturbation pair and their corresponding trajectory variations. It is known that y obyes the variational equation

$$\dot{y} = (\partial f^*/\partial x)y + (\partial f^*/\partial u)v. \tag{B.3}$$

Assume initially that $t_1^* \in (t_0,t_f)$. The other cases will be treated later. Here we use the superscript "*" to denote optimality and "^" to denote a perturbed quantity.

First we introduce two functions. Let p be a differentiable adjoint vector which satisfies

$$\dot{p} = (\partial L_i/\partial x)^T - (\partial f/\partial x)^T p, \tag{B.4}$$

where $i = 1$ on $[t_0,t_1)$ and $i = 2$ on $(t_1,t_f]$. Let the Hamiltonian $H_i(x,p,u,t)$ be defined by

$$H_i(x,p,u,t) = -L_i(x,u,t) + p^T f(x,u,t) \quad i = 1,2. \tag{B.5}$$

Using (B.5), (B.4) can be simplified to

$$\dot{p} = -(\partial H_i/\partial x)^T. \tag{B.6}$$

Now the cost corresponding to a perturbed control pair $(\hat{u},\hat{t}_1)$ is given by

$$J(\hat{u},\hat{t}_1) = \int_{t_0}^{\hat{t}_1} L_1(\hat{x},\hat{u},t)dt + \int_{\hat{t}_1}^{t_f} L_2(\hat{x},\hat{u},t)dt. \tag{B.7}$$

By adding terms which are identically zero, we get

$$\begin{aligned} J(\hat{u},\hat{t}_1) &= \int_{t_0}^{\hat{t}_1} [\hat{L}_1 + p^T(\dot{\hat{x}}-\hat{f})]dt + \int_{\hat{t}_1}^{t_f} [\hat{L}_2 + p^T(\dot{\hat{x}}-\hat{f})]dt \\ &= \int_{t_0}^{\hat{t}_1} [p^T\dot{\hat{x}}-\hat{H}_1]dt + \int_{\hat{t}_1}^{t_f} [p^T\dot{\hat{x}}-\hat{H}_2]dt. \end{aligned} \tag{B.8}$$

At each instant of time, $\hat{H}_i$ can be expanded as

$$\hat{H}_i = H_i^* + \varepsilon(\partial H_i^*/\partial x)y + \varepsilon(\partial H_i^*/\partial u)v + o(\varepsilon). \tag{B.9}$$

Substituting (B.3) and (B.9) into (B.8) yields

$$\begin{aligned} J(\hat{u},\hat{t}_1) &= \int_{t_0}^{\hat{t}_1} [p^T\dot{x}^* - H_1^* + \varepsilon\{p^T\dot{y} - (\partial H_1^*/\partial x)y - (\partial H_1^*/\partial u)v\}]dt \\ &\quad + \int_{t_1}^{t_f} [p^T\dot{x}^* - H_2^* + \varepsilon\{p^T\dot{y} - (\partial H_2^*/\partial x)y - (\partial H_2^*/\partial u)v\}]dt + o(\varepsilon) \\ &= \int_{t_0}^{t_1} [p^T\dot{x}^* - H_1^*]dt + \int_{t_1}^{t_f} [p^T\dot{x}^* - H_2^*]dt \\ &\quad - \varepsilon\int_{t_0}^{t_1} [\{\dot{p} + (\partial H_1^*/\partial x)^T\}^T y - (\partial H_1^*/\partial u)v]dt \\ &\quad - \varepsilon\int_{t_1}^{t_f} [\{\dot{p} + (\partial H_2^*/\partial x)^T\}^T y - (\partial H_2^*/\partial u)v]dt \end{aligned}$$

$$+ \varepsilon[(p_-^T \dot{x}_-^* - H_{1-}^*) - (p_+^T \dot{x}_+^* - H_{2+}^*)]\tau$$

$$+ \varepsilon[(p_-^T y_- - p_0^T y_0) + (p_f^T y_f - p_+^T y_+)] + o(\varepsilon), \qquad \text{(B.10)}$$

where the subscripts "-", "+", "0" and "f" denote the left and right limit at t_1^* and evaluation at t_0 and t_f respectively. Since $y_0 = 0$ ($x_0 = x(t_0)$ is fixed) and p satisfies (B.6), equation (B.10) can be simplified by combining the first two integrals to give $J(u^*,t_1^*)$.

$$J(\hat{u},\hat{t}_1) = J(u^*,t_1^*) - \varepsilon\left[\int_{t_0}^{t_1} (\partial H_1^*/\partial u)v\,dt + \int_{t_1}^{t_f} (\partial H_2^*/\partial u)v\,dt\right]$$

$$+ \varepsilon[p_-^T \dot{x}_-^* - p_+^T \dot{x}_+^* - H_{1-}^* + H_{2+}^*]\tau$$

$$+ \varepsilon[p_f^T y_f + p_-^T y_- - p_+^T y_+] + o(\varepsilon). \qquad \text{(B.11)}$$

Here the perturbations y_-, y_+ and τ satisfy the following relation (see Fig. A.1):

$$y_+ = y_- + (\dot{x}_-^* - \dot{x}_+^*)\tau. \qquad \text{(B.12)}$$

Using this in (B.11), we get

$$J(\hat{u},\hat{t}_1) = J(u^*,t_1^*) - \varepsilon\left[\int_{t_0}^{t_1} (\partial H_1^*/\partial u)v\,dt + \int_{t_1}^{t_f} (\partial H_2^*/\partial u)v\,dt\right]$$

$$+ \varepsilon[(p_- - p_+)^T \dot{x}_-^* + (-H_{1-}^* + H_{2+}^*)]\tau$$

$$+ \varepsilon(p_- - p_+)^T y_- + \varepsilon p_f^T y_f + o(\varepsilon). \qquad \text{(B.13)}$$

Hence the first variation δJ

$$\delta J = \lim_{\varepsilon \to 0} \frac{J(\hat{u},\hat{t}_1) - J(u^*,t_1^*)}{\varepsilon}, \qquad \text{(B.14)}$$

is given by

$$\delta J = -\int_{t_0}^{t_1} (\partial H_1^*/\partial u)v\,dt - \int_{t_1}^{t_f} (\partial H_2^*/\partial u)v\,dt + (p_- - p_+)^T y_-$$

$$+ [(p_- - p_+)^T \dot{x}_-^* + (-H_{1-}^* + H_{2+}^*)]\tau + p_f^T y_f. \tag{B.15}$$

For optimality of (u^*, t_1^*), δJ must be zero for any perturbations v, y_-, τ and y_f. Hence we finally arrive at the following set of conditions.

$$\partial H_i^*/\partial u\ (t) = 0, \quad i = 1,2, \tag{B.16}$$

$$p(t_f) = 0, \tag{B.17}$$

$$p(t_{1-}^*) = p(t_{1+}^*), \tag{B.18}$$

$$H_1^*|_{t_{1-}^*} = H_2^*|_{t_{1+}^*}. \tag{B.19}$$

When t_1^* is at one of the end points, the perturbation at t_1^* cannot be arbitrary. For example, suppose $t_1^* = t_0$ then τ must be nonnegative so that $\hat{t}_1 = t_0 + \varepsilon\tau \in [t_0, t_f]$. For this case, the first order optimality condition is given by

$$\delta J \geq 0. \tag{B.20}$$

After eliminating the terms connected with the variations in u and x (we note that the conditions (B.16) and (B.17) remain satisfied since y_f and v are arbitrary), we have

$$\delta J(\tau) = (L_{1-}^* - p_+^T f_-^* - L_{2+}^* + p_+^T f_+^*)\tau. \tag{B.21}$$

Hence we must have

$$H_1(x_0,p(t_0),u_-,t_0) \le H_2(x_0,p(t_0),u_+,t_0). \tag{B.22}$$

The other case where $t_1^* = t_f$ can be treated similarly. In fact,

$$\delta J(\tau) = (L_{1-}^* - p_-^T f_-^* - L_{2+}^* + p_-^T f_+^*)\tau, \tag{B.23}$$

with $\tau < 0$ for this case. This and (B.20) give

$$H_1(x^*(t_f),p(t_f),u_-,t_f) \ge H_2(x^*(t_f),p(t_f),u_+,t_f). \quad \| \tag{B.24}$$

Notes:

(N.1) u_- and u_+ at $t = t_0$ and t_f are solutions of

$$(\partial H_1/\partial u)(x(t),p(t),u_-,t)|_{t=t_0 \text{ or } t_f} = 0, \tag{B.25}$$

and

$$(\partial H_2/\partial u)(x(t),p(t),u_+,t)|_{t=t_0 \text{ or } t_f} = 0. \tag{B.26}$$

(N.2) If we consider $J^*(t_1) \triangleq J(u_{t_1}^*,t_1)$ to be a function of t_1 as we have done in Section 2.2, then the first order variation of J^* with respect to t_1 is actually given by

$$\delta J^*(t_1) = (-H_1^*|_{t_1-} + H_2^*|_{t_1+})\tau. \tag{B.27}$$

In fact we have

$$dJ^*(t_1)/dt_1 = -H_1^*|_{t_1-} + H_2^*|_{t_1+}. \tag{B.28}$$

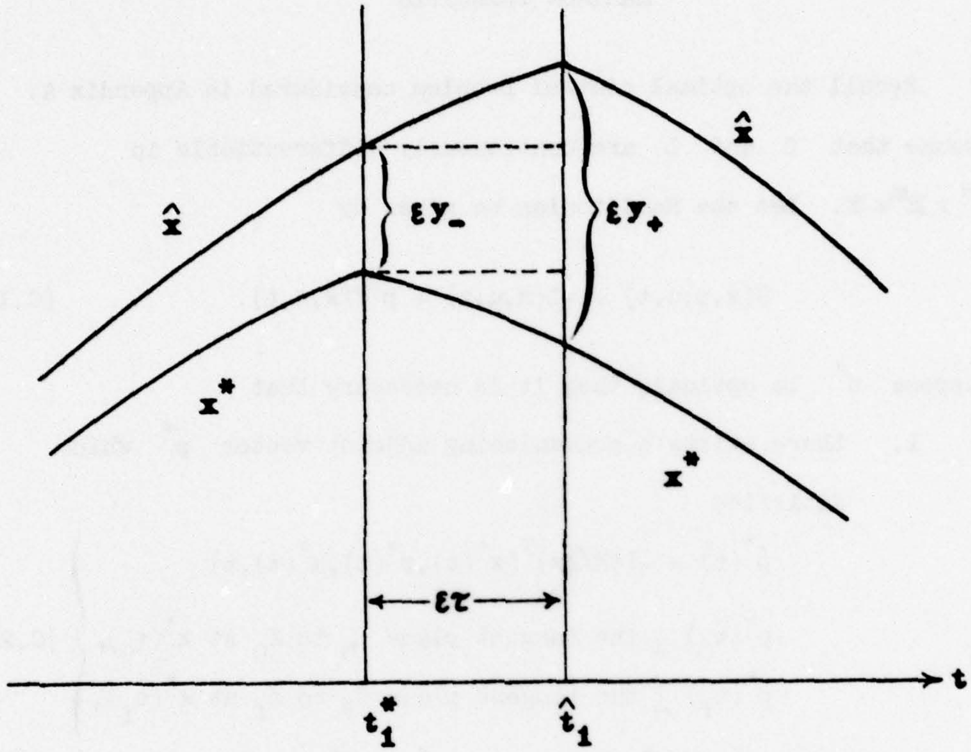

Fig. A.1. Variations τ, y_- and y_+

APPENDIX C

MAXIMUM PRINCIPLE

Recall the optimal control problem considered in Appendix A. Assume that f and L are continuously differentiable in $\mathbb{E}^n \times \mathbb{E}^m \times \mathbb{E}$. Let the Hamiltonian be given by

$$H(\mathbf{x},\mathbf{p},u,t) = -L(\mathbf{x},u,t) + \mathbf{p}^T f(\mathbf{x},u,t). \tag{C.1}$$

Suppose u^* is optimal, then it is necessary that

1. there exists a nonvanishing adjoint vector $\mathbf{p}^*$ which satisfies

$$\left.\begin{array}{l} \dot{\mathbf{p}}^*(t) = -(\partial H/\partial \mathbf{x})^T(\mathbf{x}^*(t),\mathbf{p}^*(t),u^*(t),t) \\ \mathbf{p}^*(t_0) \perp \text{the tangent plane } \Pi_0 \text{ to } X_0 \text{ at } \mathbf{x}^*(t_0), \\ \mathbf{p}^*(t_f) \perp \text{the tangent plane } \Pi_f \text{ to } X_f \text{ at } \mathbf{x}^*(t_f), \end{array}\right\} \tag{C.2}$$

2. $$\max_{u\in\Omega} H(\mathbf{x}^*(t),\mathbf{p}^*(t),u,t) = H(\mathbf{x}^*(t),\mathbf{p}^*(t),u^*(t),t) \quad \text{a.e. on } [t_0,t_f]. \tag{C.3}$$

<u>Remarks</u>:

(R.1) When the final time is free, then at an optimal final time t_f^* we have

$$H(\mathbf{x}^*(t_f^*),\mathbf{p}^*(t_f^*),u^*(t_f^*),t_f^*) = 0. \tag{C.4}$$

(R.2) When X_0 and/or X_f is reduced to a point, $\mathbf{p}^*(t_0)$ and/or $\mathbf{p}^*(t_f)$ can take on any value. When $X_f = \mathbb{E}^n$ (corresponding to free end point), $\mathbf{p}^*(t_f) = 0$.

(R.3) When the cost functional includes a terminal cost, i.e.,

$$J(u) = \int_{t_0}^{t_f} L(\mathbf{x}(t),u(t),t)dt + K(\mathbf{x}(t_f),t_f), \tag{C.5}$$

and $X_f = \mathbb{E}^n$, we have

$$\begin{aligned} \mathbf{p}^*(t_f^*) &= (\partial K/\partial \mathbf{x})(\mathbf{x}^*(t_f^*),t_f^*), \\ H^*|_{t_f^*} &= -(\partial K/\partial t_f)(\mathbf{x}^*(t_f^*),t_f^*), \end{aligned} \tag{C.6}$$

where K is assumed to be continuously differentiable in $\mathbb{E}^n \times \mathbb{E}$. ||

The maximum principle was originally proved by Pontryagin <u>et al.</u> [37].

APPENDIX D

KUHN-TUCKER CONDITIONS

A statement and discussion of the Kuhn-Tucker conditions can be found in any standard text on methematical programming. (For example, Chapter 3 of Canon, Callum and Polak [6]). We present the following Lemma without proof.

<u>Lemma D.1</u>. Consider the following nonlinear programming problem on E^m

$$\begin{cases} \text{minimize } J(u), & \\ \text{subject to } g_i(u) \leq 0, & i = 1,2,\ldots,k, \\ \qquad\qquad r_j(u) = 0, & j = 1,2,\ldots,\ell. \end{cases} \tag{D.1}$$

Let u^* solve this problem. Assume that J, g_i and r_j are differentiable at u^*. Then there exist a set of nonpositive multipliers $\lambda_0,\lambda_1,\ldots,\lambda_k$ and a set of multipliers $\mu_1,\mu_2,\ldots,\mu_\ell$, such that

1. $\lambda_i g_i(u^*) = 0, \quad i = 1,2,\ldots,k;$ (D.2)
2. $\lambda_0 \nabla f(u^*) + \sum_{i=1}^{k} \lambda_i \nabla g_i(u^*) + \sum_{j=1}^{\ell} \mu_j \nabla r_j(u^*) = 0.$ (D.3)

||

APPENDIX E

COMPUTATION OF GRADIENT $\nabla_{\underline{u}} M(\underline{u})$

Recall that $\nabla_{\underline{u}} M(\underline{u})$ has three terms

$$\nabla_{\underline{u}} M(\underline{u}) = \lambda_0 \nabla_{\underline{u}} J(\underline{u}) + \mu \nabla_{\underline{u}} h(x_k(\underline{u})) + \sum_{i=1}^{k} \lambda_i \nabla_{\underline{u}} g_i(\underline{u}). \tag{E.1}$$

We will calculate each term separately. First consider $\nabla_{\underline{u}} g_i(\underline{u})$.

From (IV.2.3), we get

$$\frac{\partial g_j}{\partial u_i}(\underline{u}) = \left\{ \begin{array}{ll} 0 & , \; i \neq j \\ 2\{u_i - (v_i + w_i)/2\}, & i = j \end{array} \right\} . \tag{E.2}$$

Hence

$$\sum_{i=1}^{k} \lambda_i \nabla_{\underline{u}} g_i(\underline{u}) = \begin{bmatrix} 2\lambda_1(u_1 - \bar{u}_1) \\ 2\lambda_2(u_2 - \bar{u}_2) \\ \vdots \\ 2\lambda_k(u_k - \bar{u}_k) \end{bmatrix} , \tag{E.3}$$

where $\bar{u}_i = (v_i + w_i)/2, \;\; i = 1,2,\ldots,k.$

Next compute $\nabla_u h(x_k(\underline{u}))$. Let $x_i \triangleq x(t_i), \; i = 1,2,\ldots,k,$ then

$$\begin{aligned} (\partial h(x_k(\underline{u}))/\partial u_i) &= (\partial h(x_k)/\partial x_k)(\partial x_k/\partial u_i) \\ &= (\partial h(x_k)/\partial x_k)(\partial x_k/\partial x_{k-1})\ldots(\partial x_{i+1}/\partial x_i)(\partial x_i/\partial u_i) \end{aligned} \tag{E.4}$$

Let $x(t;t_{i-1},x_{i-1},u_i)$ be the solution trajectory of the system equation (IV.1.1) on a subinterval $[t_{i-1},t_i)$ such that

$$\left\{\begin{array}{l}\dot{x}(t;t_{i-1},x_{i-1},u_i) = f(x(t;t_{i-1},x_{i-1},u_i),u_i),\\ x(t_{i-1};t_{i-1},x_{i-1},u_i) = x_{i-1}.\end{array}\right\} \tag{E.5}$$

Then $x(t;t_{i-1},x_{i-1},u_i)$ on $[t_{i-1},t_i]$ can be written as

$$x(t;t_{i-1},x_{i-1},u_i) = x_{i-1} + \int_{t_{i-1}}^{t} f(x(\tau;t_{i-1},x_{i-1},u_i),u_i)d\tau. \tag{E.6}$$

By differentiating both sides by u_i, we get

$$\begin{aligned}&\partial x(t;t_{i-1},x_{i-1},u_i)/\partial u_i\\ &= \int_{t_{i-1}}^{t} [\{(\partial f/\partial x)(x(\tau;t_{i-1},x_{i-1},u_i),u_i)\}\{(\partial x/\partial u_i)(\tau;t_{i-1},x_{i-1},u_i)\}\\ &\qquad + (\partial f/\partial u_i)(x(\tau;t_{i-1},x_{i-1},u_i),u_i)]d\tau.\end{aligned} \tag{E.7}$$

This shows that $\partial x(t)/\partial u_i$ on $[t_{i-1},t_i)$ satisfies the initial value problem

$$\left.\begin{array}{l}\{d(\partial x/\partial u_i)/dt\}(t) = f_x(x(t),u_i)\{(\partial x/\partial u_i)(t)\} + f_u(x(t),u_i),\\ (\partial x/\partial u_i)(t_{i-1}) = 0,\end{array}\right\} \tag{E.8}$$

where the subscripts x and u denote partial differentiation with respect to x and u respectively. Hence on $[t_{i-1},t_i]$

$$(\partial x/\partial u_i)(t) = \int_{t_{i-1}}^{t} \Phi(t,\tau)f_u(x(\tau),u_i)d\tau, \tag{E.9}$$

where $\Phi(t,\tau)$ is the state transition matrix of the system (E.8), i.e.,

$$\left.\begin{array}{l}d\Phi(t,\tau)/dt = f_x(x(t),u_i)\Phi(t,\tau),\\ \Phi(\tau,\tau) = I.\end{array}\right\} \tag{E.10}$$

Therefore $\partial x_i/\partial u_i$ is given by

$$\partial x_i/\partial u_i = \int_{t_{i-1}}^{t_i} \Phi(t_i,\tau)f_u(x(\tau),u_i)d\tau. \tag{E.11}$$

Now, by differentiating (E.6) with respect to x_{i-1} and replacing i with j, we have for $t \in [t_{j-1},t_j)$

$$(\partial x/\partial x_{j-1})(t) - I = \int_{t_{j-1}}^{t} f_x(x(\tau),u_i)\{(\partial x/\partial x_{j-1})(\tau)\}d\tau. \tag{E.12}$$

Again this shows that $(\partial x/\partial x_{j-1})(t)$ is a solution of

$$\left.\begin{aligned} \{d(\partial x/\partial x_{j-1})/dt\}(t) &= f_x(x(t),u_j)\{(\partial x/\partial x_{j-1})(t)\}, \\ (\partial x/\partial x_{j-1})(t_{j-1}) &= I. \end{aligned}\right\} \tag{E.13}$$

By comparing (E.13) with (E.10) we immediately have

$$\partial x_j/\partial x_{j-1} = (\partial x/\partial x_{j-1})(t_j) = \Phi(t_j,t_{j-1}). \tag{E.14}$$

Using (E.9) and (E.14), (E.4) can be written as

$$(\partial h/\partial u_i)(x_k(\underline{u})) =$$

$$= (\partial h/\partial x_k)\Phi(t_k,t_{k-1})\ldots\Phi(t_{i+1},t_i)\int_{t_{i-1}}^{t_i} \Phi(t_i,\tau)f_u(x(\tau),u_i)d\tau$$

$$= (\partial h/\partial x_k)\int_{t_{i-1}}^{t_i} \Phi(t_k,\tau)f_u(x(\tau),u_i)d\tau\ . \tag{B.15}$$

This gives

$$\mu\nabla_{\underline{u}} h(x_k(\underline{u})) = \begin{bmatrix} \mu(\partial h(x_k)/\partial x_k)\int_{t_1}^{t_0} \Phi(t_k,\tau)f_u(x(\tau),u_1)d\tau \\ \mu(\partial h(x_k)/\partial x_k)\int_{t_1}^{t_2} \Phi(t_k,\tau)f_u(x(\tau),u_2)d\tau \\ \vdots \\ \mu(\partial h(x_k)/\partial x_k)\int_{t_{k-1}}^{t_k} \Phi(t_k,\tau)f_u(x(\tau),u_k)d\tau \end{bmatrix} . \qquad \text{(E.16)}$$

Now we proceed to the final term $\nabla_{\underline{u}} J(\underline{u})$. Since $x(t)$, $t \leq t_i$ does not depend on u_i, we have

$$\begin{aligned}(\partial J/\partial u_i)(\underline{u}) &= \partial\{\sum_{j=1}^{k}\int_{t_{j-1}}^{t_j} L_j(x(\tau),u_j)d\tau\}/\partial u_i \\ &= \sum_{j=i}^{k}\int_{t_{j-1}}^{t_j} [L_{jx}(x(\tau),u_j)\{(\partial x/\partial u_i)(\tau)\} \\ &\quad + L_{ju}(x(\tau),u_j)\ (\partial u_j/\partial u_i)\ ]d\tau \\ &= \int_{t_{i-1}}^{t_i} L_{iu}(x(\tau),u_i)d\tau \\ &\quad + \sum_{j=i}^{k}\int_{t_{j-1}}^{t_j} L_{jx}(x(\tau),u_j)\{(\partial x/\partial u_i)(\tau)\}d\tau \end{aligned} \qquad \text{(E.17)}$$

Again by (E.9) and (E.14), for $\tau \geq t_i$

$$(\partial x/\partial u_i)(\tau) = \int_{t_{i-1}}^{t_i} \Phi(\tau,s)f_u(x(s),u_i)ds. \qquad \text{(E.18)}$$

Substituting (E.9) and (E.18) into (E.17), we get

$$\begin{aligned}(\partial J/\partial u_i)(\underline{u}) &= \int_{t_{i-1}}^{t_i} [L_{iu}(x(\tau),u_i) \\ &\quad + L_{ix}(x(\tau),u_i)\int_{t_{i-1}}^{\tau} \Phi(\tau,s)f_u(x(s),u_i)ds\ ]d\tau \\ &\quad + \sum_{j=i+1}^{k}\int_{t_{j-1}}^{t_j} L_{jx}(x(\tau),u_j)\int_{t_{i-1}}^{t_i} \Phi(\tau,s)f_u(x(s),u_i)ds\,d\tau. \end{aligned}$$

(E.19)

By interchanging the order of integration and rearranging terms, (E.19) can be expressed as

$$\partial J(\underline{u})/\partial u_i = \int_{t_{i-1}}^{t_i} [\sum_{j=i+1}^{k} \int_{t_{j-1}}^{t_j} L_{jx}(x(\tau),u_j)\Phi(\tau,s)d\tau]f_u(x(s),u_i)ds$$

$$+ \int_{t_{i-1}}^{t_i} [L_{iu}(x(s),u_i) + \int_s^{t_i} L_{ix}(x(\tau),u_i)\Phi(\tau,s)d\tau \; f_u(x(s),u_i)]ds. \tag{E.20}$$

For notational simplicity, let

$$L(x,u,t) \triangleq L_i(x,u_i) \text{ on } [t_{i-1},t_i),\; i = 1,2,\ldots,k, \tag{E.21}$$

then

$$\partial J(\underline{u})/\partial u_i = \int_{t_{i-1}}^{t_i} [L_u(x(t),u,t) + \int_t^{t_k} L_x(x(\tau),u,\tau)\Phi(\tau,t)d\tau f_u(x(t),u_i)]dt. \tag{E.22}$$

Combining (E.15) and (E.22) leads to

$$\partial\tilde{J}(\underline{u})/\partial u_i = \lambda_0(\partial J(\underline{u})/\partial u_i) + \mu(\partial h(x_k(\underline{u}))/\partial u_i)$$

$$= \int_{t_{i-1}}^{t_i} [\Phi^T(t_k,t)\mu(\partial h(x_k)/\partial x_k)^T$$

$$+ \lambda_0 \int_t^{t_k} \Phi^T(\tau,t)L_x^T(x(\tau),u,\tau)d\tau]^T f_u(x(t),u_i)dt$$

$$+ \int_{t_{i-1}}^{t_i} \lambda_0 L_u(x(t),u,t)dt. \tag{E.23}$$

Now let p be an adjoint vector satisfying

$$\left\{\begin{aligned} \dot{p}(t) &= -f_x^T(x(t),u(t))p(t) - \lambda_0 L_x^T(x(t),u(t),t), \\ p(t_k) &= \mu(\partial h(x_k)/\partial x_k)^T. \end{aligned}\right\} \tag{E.24}$$

Equation (E.24) can be solved to yield

$$p(t) = \psi(t,t_k)p(t_k) + \int_t^{t_k} \psi(t,\tau)\lambda_0 L_x^T(x(\tau),u,\tau)d\tau, \tag{E.25}$$

where $\psi(t,\tau)$ is the state transition matrix for the adjoint system (E.24), i.e.

$$\left.\begin{aligned} \frac{d}{dt}\,\psi(t,\tau) &= -f_x^T(x(t),u(t))\psi(t,\tau), \\ \psi(\tau,\tau) &= I. \end{aligned}\right\} \tag{E.26}$$

It is well-known that

$$\psi(t,\tau) = (\Phi^{-1}(t,\tau))^T = \Phi^T(\tau,t). \tag{E.27}$$

Now, combining (E.23), (E.24), (E.25) and (E.27),

$$\partial\tilde{J}(\underline{u})/\partial u_i = \int_{t_{i-1}}^{t_i} [\lambda_0 L_u(x(t),u,t) + p^T(t)f_u(x(t),u_i)]dt. \tag{E.28}$$

We can further simplify (E.28) by introducing the Hamiltonian

$$H(x,p,u,\lambda_0,t) = \lambda_0 L(x,u,t) + p^T f(x,u). \tag{E.29}$$

Then

$$\partial\tilde{J}(\underline{u})/\partial u_i = \int_{t_{i-1}}^{t_i} (\partial H/\partial u)(x(t),p(t),u(t),\lambda_0,t)dt. \tag{E.30}$$

Note that since $L(x,u,t) = L_i(x,u_i)$ on $[t_{i-1},t_i)$, we also can write

$$\partial\tilde{J}(\underline{u})/\partial u_i = \int_{t_{i-1}}^{t_i} (\partial H_i/\partial u_i)(x(t),p(t),u_i,\lambda_0)dt, \tag{E.31}$$

by defining

$$H_i(x,p,u_i,\lambda_0) = \lambda_0 L_i(x,u_i) + p^T f(x,u_i). \tag{E.32}$$

Thus, we have

$$\nabla_{\underline{u}} \tilde{J}(\underline{u}) = \begin{bmatrix} \int_{t_0}^{t_1} (\partial H_1/\partial u_1)(x(t),p(t),u_1,\lambda_0)dt \\ \int_{t_1}^{t_2} (\partial H_2/\partial u_2)(x(t),p(t),u_2,\lambda_0)dt \\ \vdots \\ \int_{t_{k-1}}^{t_k} (\partial H_k/\partial u_k)(x(t),p(t),u_k,\lambda_0)dt \end{bmatrix} . \tag{E.33}$$

Finally, combining (E.33) and (E.3), we have the desired gradient $\nabla_{\underline{u}} M(\underline{u}^*)$.

$$\nabla_{\underline{u}} M(\underline{u}^*) = \begin{bmatrix} \int_{t_0}^{t_1} (\partial H_1^*/\partial u_1)(t)dt + 2\lambda_1(u_1^* - \bar{u}_1) \\ \int_{t_1}^{t_2} (\partial H_2^*/\partial u_2)(t)dt + 2\lambda_2(u_2^* - \bar{u}_2) \\ \vdots \\ \int_{t_{k-1}}^{t_k} (\partial H_k^*/\partial u_k)(t)dt + 2\lambda_k(u_k^* - \bar{u}_k) \end{bmatrix} , \tag{E.34}$$

where

$$(\partial H_i^*/\partial u_i)(t) = (\partial H_i/\partial u_i)(x^*(t),p^*(t),u_i^*,\lambda_0^*). \tag{E.35}$$

APPENDIX F

COEFFICIENTS OF THE TWO-TEMPERATURE MODEL

The coefficients A_j, B_k and C_ℓ of the two-temperature model (V.1.2a) - (V.1.2d) are positive constants given by

$$\left.\begin{array}{ll}
A_1 = 3.7\times 10^2 \gamma_J Z_{eff}(a^4 N_e^2 \tilde{Z})^{-1}, & A_2 = \hat{A}_2(N_e\tilde{Z})^{-1}, \\
A_3 = 0.148 Z_{eff}(AZ_n\tilde{Z})^{-1}, & A_4 = 0.116 N_I(N_e\tilde{Z})^{-1}, \\
A_5 = 1.86 C_1(N_e\tilde{Z})^{-1}, & B_1 = 6.05\times 10^{-4}(a/R)^2 A^{1/2} Z_{eff}(Z_n\tilde{Z})^{-1}, \\
B_2 = 1.11 A^{1/2}(N_e Z_n \tilde{Z} B R^2)^{-1}, & B_3 = 5.2\times 10^{-3}(a/R)^{1/2} A^{1/2} Z_{eff}(Z_n\tilde{Z})^{-1}, \\
B_4 = 0.186 N_o(N_e Z_n\tilde{Z})^{-1}, & B_5 = 1.86 C_2(N_e\tilde{Z})^{-1}, \\
C_1 = 0.56 Z_N^2(A_N\tilde{Z})^{-1}, & C_2 = A_n^{1/2} Z_n
\end{array}\right\} \tag{F.1}$$

with

$$N_e Z_{eff} = \sum_j N_j Z_j^2, \quad N_e\tilde{Z} = \sum_j N_j \tilde{A}_j^{-1} Z_j^2, \quad N_e = Z_n N_i, \quad N_I = \sum_{j\neq i} N_j, \tag{F.2}$$

where A and Z_i (resp. A_N and Z_N) are the mass number and ionic charge number of the ions (resp. injected neutral particles) respectively; $\tilde{A}_j$ and Z_j are the mass and the ionic charge numbers of the j-th species in the plasma or the impurities; N_e, N_i, N_o and N_j are respectively the number densities of the electrons, ions, neutrals and the ions of the j-th species in the impurities; Z_{eff} and $\tilde{Z}$ are respectively the effective and the modified effective charge numbers; B is the toroidal magnetic field; a and R are the minor and major radii of the plasma torus respectively, and γ_J is the correction

factor for the Joule heating term (including trapped-particle effects and Z-correction). The coefficient $\hat{A}_2$ depends on the regime of electron diffusion. In the collisional regime $(T_e < T_e^c)$, $\alpha = 1/2$, $\beta = 0$ and $\hat{A}_2 = \gamma_e Z_{eff} N_e/(aB)^2$, where γ_e is an anomaly factor, and T_e^c is the transition temperature for this regime. In the regime $(T_e > T_e^c)$ where the electron-loss is dominated by trapped-electron instability, we have $\alpha = 13/2$, $\beta = 4$ and $\hat{A}_2 = \gamma_e N_e/(B^2 Z_{eff} \tilde{\varepsilon}^{3/2} \beta_{pe}^2)$, where $\tilde{\varepsilon} = a/R$ and $\beta_{pe} = 0.3$ is the "poloidal beta" associated with the electrons.

For the T.F.R. experiments, the specific values for various parameters are: $\gamma_J = 2$, $A = 1$, $Z_i = 1$, $a = 2$, $R = 1$, $Z_n = 2$, $B = 4$, $Z_N = 1$, $A_N = 1$, $Z_{eff} = 4$, $\tilde{Z} = 1/2$, $N_i = 2$, $N_I = 3$ and $N_o = 5$, where the units are as follows: T_e, T_i (KeV), E(10 KeV), I (10 amperes), I_p (10^6 amperes), N_e, N_i ($10^{13}/cm^3$), N_o ($10^8/cm^3$), N_I ($10^{11}/cm^3$), a (10 cm), R (100 cm), and B (10^4 gauss).

The two-temperature model is formulated on the normalized time scale τ which is related to the real time scale t(m sec) by $d\tau = (28AN_i^{-1}Z_i^{-2})dt$.

The constant R in the cost functional (V.1.4) is given by

$$R = 1.86A/(Z^2 N_i). \tag{F.3}$$

APPENDIX G

HAMILTONIANS AND ADJOINT EQUATIONS FOR PLASMA HEATING PROBLEM

The Hamiltonians (V.3.1) and (V.3.2) in Chapter V are given in explicit form as follows:

$$\begin{aligned}
H_2(x,p,I,t) = {} & (Ep_3T_e^{-3/2} + E^{-1/2}p_4 - RE)I - A_1I_p^2T_e^{-3/2} \\
& + p_1\{A_1I_p^2T_e^{-3/2} - A_2I_p^{-\beta}T_e^{\alpha} - A_3(T_e - T_i)T_e^{-3/2} - A_4 + A_5X_1\} \\
& + p_2\{A_3(T_e - T_i)T_e^{-3/2} - S_{Di}(T_i;I_p) - B_4T_i + B_5X_2\} \\
& + p_3\{(C_1T_e^{-3/2} + C_2E^{-3/2})X_1\} \\
& + p_4\{(C_1T_e^{-3/2} + C_2E^{-3/2})X_2\},
\end{aligned} \tag{G.1}$$

$$H_1(x,p,I,t) = H_2(x,p,I,t) + REI. \tag{G.2}$$

The adjoint equations (V.3.10) are written explicitly below in equation (G.3). Note that since the Jacobians $(\partial H_1/\partial x)$ and $(\partial H_2/\partial x)$ are identical, the adjoint vector $p(t)$ satisfies the same differential equation (G.3) on two stages.

$$\begin{aligned}
\dot{p}_1 = {} & -\tfrac{3}{2}A_1I_p^2T_e^{-5/2} \\
& + p_1\{\tfrac{3}{2}A_1I_p^2T_e^{-5/2} + \alpha A_2I_p^{-\beta}T_e^{\alpha-1} - \tfrac{1}{2}A_3(T_e - 3T_i)T_e^{-5/2}\} \\
& + p_2\{\tfrac{1}{2}A_3(T_e - 3T_i)T_e^{-5/2}\} + p_3\{\tfrac{3}{2}T_e^{-5/2}EI - \tfrac{3}{2}C_1T_e^{-5/2}X_1\} \\
& + p_4\{-\tfrac{3}{2}C_1X_2\}, \\
\dot{p}_2 = {} & p_1\{-A_3T_e^{-3/2}\} + p_2\{A_3T_e^{-3/2} + \partial S_{Di}(T_i;I_p)/\partial T_i + B_4\}, \\
\dot{p}_3 = {} & p_1\{-A_5\} + p_3\{C_1T_e^{-3/2} + C_2E^{-3/2}\}, \\
\dot{p}_4 = {} & p_2\{-B_5\} + p_4\{C_1T_e^{-3/2} + C_2E^{-3/2}\}.
\end{aligned} \tag{G.3}$$

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